# COUNTABLE MARKOV SHIFTS WITH TRANSIENT POTENTIALS

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Abstract. We define a simple property on an infinite directed graph $\mathcal{G}$ and show that it is necessary and sufficient for the existence of a transient potential on the associated countable Markov shift.

## 1. Introduction

1.1. **Motivation.** One of the main problems in thermodynamic formalism is, given a dynamical system $T : X \to X$ and a function $\phi : X \to \mathbb{R}$ (called the *potential*), to determine the existence and uniqueness of invariant probability measures maximizing the quantity

$$h_\mu(T) + \int \phi d\mu.$$

If $X$ is a countable Markov shift, the supremum of the above quantity over all invariant probability measures is equal to the *Gurevich pressure* (see [10]) and is denoted $P_G(\phi)$ (see next section for definition). A solution to the above variational problem (i.e. an invariant probability measure with $h_\mu(T) + \int \phi d\mu = P_G(\phi)$) is called an *equilibrium measure* (such a measure minimizes the "free energy" of the system).

A related question is to study how equilibrium measures change along one-parameter families of potentials $\{\phi+t\cdot\psi\}_{t\in\mathbb{R}}$. Apart from existence and uniqueness results, this requires us to study the regularity properties of the pressure $P_G(\phi+t\cdot\psi)$. If $X$ is a topologically mixing *finite* state Markov shift, and $\phi, \psi$ are Hölder continuous, then a theorem of Ruelle [7] states that the pressure is real-analytic on some $t$-neighborhood of zero. For *countable* Markov shifts this may not be the case (see [12, 13]). For $\phi$ such that $P_G(\phi+t\cdot\psi)$ is not analytic at $t = 0$, the system $(X, T, \phi)$ is said to undergo a *phase transition*.

The topic of this paper is the existence of potentials on a countable Markov shift that exhibit a phenomenon known as *transience* (see next section). The interest in transient potentials is twofold:

- Transience is an obstruction to the existence of an equilibrium measure [3];
- If $\phi$ is transient and $a \in \mathcal{S}$ (see next section) then there is some $s_0 > 0$ so that $P_G(\phi + s \cdot 1_{[a]}) = P_G(\phi)$ for all $s \le s_0$ and $P_G(\phi + s \cdot 1_{[a]}) > P_G(\phi)$ for all $s > s_0$ [4, Theorem 6.1]; that is, the family $\{\phi + s \cdot 1_{[a]}\}$ exhibits a phase transition at $s = s_0$.

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*Date*: March 31, 2009.

2000 *Mathematics Subject Classification.* 37D35.

*Key words and phrases.* Thermodynamic formalism, transience, Rome, phase transition.

The author was partially funded by NSF Grant DMS-0652966.

The main result of this paper is a (checkable) criterion on a graph $\mathcal{G}$ which is necessary and sufficient for the associated countable Markov shift $X_{\mathcal{G}}$ to have a transient potential. This condition is very mild, so we can say that "most" countable Markov shifts have at least one transient potential.

1.2. **Background.** Let $\mathcal{G}$ be a directed graph with countable vertex set $\mathcal{S}_{\mathcal{G}}$. If $v_1, v_2 \in \mathcal{S}_{\mathcal{G}}$ we denote a directed edge from $v_1$ to $v_2$ by $v_1 \to v_2$. The *countable Markov shift* associated to $\mathcal{G}$ is the dynamical system on the space of one-sided infinite trajectories through $\mathcal{G}$, denoted by

$$X_{\mathcal{G}} := \{(x_0, x_1, x_2, \dots) \mid x_i \in \mathcal{S}_{\mathcal{G}} \text{ and } x_i \to x_{i+1}\},$$

whose dynamics are given by the left-shift $T : X_{\mathcal{G}} \to X_{\mathcal{G}}$,

$$T(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots).$$

$X_{\mathcal{G}}$ is given the metric $d(x, y) = 2^{-\min\{i | x_i \neq y_i\}}$ and this topology is generated by the *cylinder sets*

$$[a_0, a_1, \dots, a_{n-1}] := \{x \in X_{\mathcal{G}} \mid x_i = a_i,\ 0 \leq i < n\}.$$

A shift $X_{\mathcal{G}}$ is *topologically mixing* if for any $a, b \in \mathcal{S}_{\mathcal{G}}$ there is an integer $N(a, b)$ so that for any $n > N(a, b)$ there is a path from $a$ to $b$ of length $n$.

For a potential $\phi : X_{\mathcal{G}} \to \mathbb{R}$ define

$$var_n(\phi) := \sup\{|\phi(x) - \phi(y)| : x_i = y_i \text{ for all } 0 \leq i < n\}.$$

We say that $\phi$ has *summable variations* if $\sum_{n=2}^{\infty} var_n(\phi) < \infty$. A potential is *weakly Hölder continuous* if there exist $\theta \in (0, 1)$ and $A > 0$ s.t. $var_n(\phi) \leq A \cdot \theta^n$ for all $n \geq 2$. We define

$$\Phi := \{\phi : X_{\mathcal{G}} \to \mathbb{R} \mid \phi \text{ is weakly Hölder continuous}\}.$$

We set $\phi_n(x) = \phi(x) + \phi(Tx) + \cdots + \phi(T^{n-1}x)$ and define the *Gurevich pressure* of $\phi$ to be

$$P_G(\phi) := \lim_{n\to\infty} \frac{1}{n} \log(Z_n(\phi, a)), \text{ where } Z_n(\phi, a) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x).$$

If $X_{\mathcal{G}}$ is topologically mixing and $\phi$ has summable variations, then the limit defining $P_G(\phi)$ exists and is independent of $a$ ([10], also see [11], [12] for an overview of thermodynamic formalism for countable Markov shifts).

For $\phi \in \Phi$, we define the generating function of the sequence $Z_n(\phi, a)$ to be

$$t_\phi(x) = 1 + \sum_{n=1}^{\infty} Z_n(\phi, a) x^n$$

and note that its radius of convergence is $e^{-P_G(\phi)}$. We say that $\phi \in \Phi$ is

- *recurrent* if $t_\phi(e^{-P_G(\phi)}) = \infty$;
- *transient* if $t_\phi(e^{-P_G(\phi)}) < \infty$.

It was shown in [4] that $\phi$ is transient if and only if there exists $\psi > 0$ so that $P_G(\phi) = P_G(\phi + \psi)$.

A potential $\phi(x)$ is called a *Markov potential* if $\phi(x) = \phi(x_0, x_1)$ (i.e. it depends on only the first two coordinates of $x = (x_0, x_1, \dots)$). When $\phi$ is a Markov potential, determining the convergence of $t_\phi(e^{-P_G(\phi)})$ is usually done by analyzing $Z_n^*(\phi, a) := \sum_{T^n x = x} e^{\phi_n(x)} 1_{[\varphi_a = n]}(x)$ where $\varphi_a(x) = 1_{[a]}(x) \inf\{n \geq 1 \mid T^n x \in [a]\}$. We note that (for Markov potentials) $Z_n = Z_n^* + Z_{n-1}^* Z_1 + \cdots + Z_1^* Z_{n-1}$. In this case, we take

$$r_\phi(x) = \sum_{n=1}^{\infty} Z_n^*(\phi, a)$$

and note that $t_\phi(x) - 1 = t_\phi(x) r_\phi(x)$ or more simply

$$t_\phi(x) = \frac{1}{1 - r_\phi(x)}. \tag{1.1}$$

Note that, by definition of $Z_n^*(\phi, a)$, $r_{\phi + t\cdot 1_{[a]}}(x) = e^t r_\phi(x)$ so if $r_\phi(x)$ converges at its radius of convergence, there is an $s$ so that $t_{\phi - s\cdot 1_{[a]}}(x)$ converges at its radius of convergence. We restate this as the following (frequently used) lemma:

**Lemma 1.1.** *Suppose $\phi$ is a Markov potential on $X_\mathcal{G}$ for which $r_\phi(x)$ converges at its radius of convergence. Then there is a transient Markov potential on $X_\mathcal{G}$.*

It was shown in [4] that if $\phi$ is a transient potential on $X_\mathcal{G}$ and $\mathcal{F} \subset \mathcal{S}_\mathcal{G}$ is a *finite* set, then there is some $t_0 = t_0(\mathcal{F}) > 0$ such that $\phi + t \cdot 1_\mathcal{F}$ is transient with pressure $P_G(\phi)$ for all $0 \le t < t_0$. Therefore if there is a transient potential on $X_\mathcal{G}$ then there are infinitely many.

1.2.1. *Graph Theoretic Notation.* We will need the following definitions frequently.

If there is a path $p$ in $\mathcal{G}$ from $v_1$ to $v_2$ with $n$ vertices (including $v_1$, $v_2$), we will write

$$p : v_1 \xrightarrow{n} v_2.$$

The *length* of a path is the number $n$ above and will be denoted by $L(p)$. We will denote the $i^{th}$ vertex in the path $p$ by $(p)_i$ where $(p)_0$ is the initial vertex and $(p)_{L(p)-1}$ is the terminal vertex. Also, we will denote by $V(p)$ the set of all vertices in the path $p$.

Often we will need to refer to paths that have no repeated vertices. We will call such a path *simple*. A loop is a path $p$ with $(p)_0 = (p)_{L(p)-1}$. Analogous to a simple path, we will say that a loop is *simple* if $(p)_i = (p)_j$ implies either that $i = j$ or that $i = 0$ and $j = L(p) - 1$.

The following is a new definition which is needed to formulate our main theorem.

**Definition 1.1.** *Let $\mathcal{G}$ be a directed graph. A subset $F \subseteq \mathcal{S}_\mathcal{G}$ is a* uniform Rome[1] *for $\mathcal{G}$ if there exists a positive integer $N$ such that the graph $\mathcal{G} \setminus F$ [2] has no paths of length greater than $N$. A* finite uniform Rome *is a uniform Rome with $|F| < \infty$.*

## 2. Main Results

Our main result is the following:

**Theorem 2.1.** *The set of transient potentials on the shift $X_\mathcal{G}$ is nonempty if and only if the graph $\mathcal{G}$ does* not *have a finite uniform Rome.*

For several examples of graphs with finite uniform Romes, see the Appendix A.

[1] The term *Rome* first appeared in [1]. ("All roads lead to Rome.")

[2] $\mathcal{G} \setminus F$ is the graph obtained by deleting all vertices in $F$ from $\mathcal{G}$ as well as all directed edges to or from these vertices.

2.1. **Overview of Proof of Theorem 2.1.** The proof has three steps.

(1) Show that if $\mathcal{G}$ has a finite uniform Rome then any potential in $\Phi$ with finite pressure must be recurrent. [Section 3]
(2) Show that for several large classes of graphs, the associated shift has a transient potential. [Sections 4 and 5]
(3) Show that if a graph is not in one of these classes then it has a finite uniform Rome. [Section 6]

We now briefly elaborate on the ideas that go into each of these steps.

*First Step.* This is a straightforward consequence of the definition of a finite uniform Rome and the following two theorems which appear in [4]:

**Theorem 2.2** (Theorem 2.3 in [4]). *Let $\Phi(T)$ denote the set of transient $\phi \in \Phi$. Let $\mathcal{LU}(F)$ denote the topology generated by sets of the form*

$$U(\phi;\varepsilon,F) := \{\phi' \in \Phi : \|\phi' - \phi\|_\infty < \varepsilon, \phi'|_{X_{\mathcal{G}}\setminus F} = \phi|_{X_{\mathcal{G}\setminus F}}\} \ (\varepsilon > 0).$$

*Then, with respect to $\mathcal{LU}(F)$, $\Phi(T)$ is open in $\Phi$.*

**Theorem 2.3** (Theorem 4.1 in [4]). *If $\phi \in \Phi$ is a transient potential, $a \in \mathcal{S}$ and $\psi$ is a non-positive bounded function with summable variations s.t. supp$(\psi) \subset [a]$, then $\phi + \psi \in \Phi$, $\phi + \psi$ is transient and $P_G(\phi + \psi) = P_G(\phi)$.*

*Second Step.* This requires the most effort. The main tool and starting point for this step is the following:

**Theorem 2.4** (Extension Theorem). *If $\mathcal{H} \subseteq \mathcal{G}$ is a topologically mixing subgraph and $\phi = \phi(x_0, x_1)$ is a transient Markov potential on $X_{\mathcal{H}}$, then $\phi$ can be extended to a transient Markov potential $\widetilde{\phi}$ on $X_{\mathcal{G}}$ with $P_G(\widetilde{\phi}) = P_G(\phi)$.*

Thus showing that a particular shift $X_{\mathcal{H}}$ has a transient potential in fact shows that $X_{\mathcal{G}}$ has a transient potential for any $\mathcal{G}$ containing $\mathcal{H}$ as a subgraph. Thus is suffices to explicitly find a transient potential for several (fairly simple) shifts and then conclude that any graph containing one of these as a subshift also has a transient potential. Specifically we show:

- If $\mathcal{G}$ contains an infinite forward or backward ray[3], then $X_{\mathcal{G}}$ has a transient potential.
- If $\mathcal{G}$ has infinitely many disjoint simple loops (based at different base points), then $X_{\mathcal{G}}$ has a transient potential.
- If there is a vertex $a \in \mathcal{S}_{\mathcal{G}}$ such that there are simple loops based at $a$ of arbitrarily large size, then $X_{\mathcal{G}}$ has a transient potential.

*Third Step.* We assume that $\mathcal{G}$ does not have any of the attributes in the bulleted list above and explicitly find a finite uniform Rome.

[3]An *infinite forward ray* is an infinite simple path of the form:

$$v_1 \to v_2 \to v_3 \to \cdots$$

An *infinite backward ray* is an infinite simple path of the form:

$$v_1 \leftarrow v_2 \leftarrow v_3 \leftarrow \cdots$$

## 3. A Finite Uniform Rome implies Recurrence

**Theorem 3.1.** *If $\mathcal{G}$ has a finite uniform Rome then every weakly Hölder potential on $X_{\mathcal{G}}$ with finite pressure is recurrent.*

*Proof.* Suppose $\mathcal{G}$ has a finite uniform Rome $F$ with parameter $N$ and $\phi \in \Phi$ is a potential on $X_{\mathcal{G}}$ with finite pressure. We claim that $\phi$ is recurrent. For any $\varepsilon > 0$,

$$\begin{aligned} Z_n(\phi + \varepsilon \cdot 1_F, a) &= \sum_{T^n x = x} e^{(\phi+\varepsilon \cdot 1_F)_n(x)} 1_{[a]}(x) \\ &\geq e^{\lfloor n \cdot \varepsilon / N \rfloor} \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x) \\ &= e^{\lfloor n \cdot \varepsilon / N \rfloor} Z_n(\phi, a) \end{aligned}$$

so that $P_G(\phi + \varepsilon \cdot 1_F) \geq P_G(\phi) + \frac{\varepsilon}{N} > P_G(\phi)$. If $\phi$ were transient, then for all sufficiently small $\varepsilon$ we would have $\phi + \varepsilon \cdot 1_F$ transient (Theorem 2.2). But then $P_G(\phi + \varepsilon \cdot 1_F) = P_G(\phi)$ (repeatedly apply Theorem 2.3 with $\psi = -\varepsilon \cdot 1_{[b]}$ for each $b \in F$); a contradiction. So $\phi$ is recurrent. □

## 4. Proof of The Extension Theorem

Our goal in this section is to prove theorem 2.4. Starting with a transient Markov potential $\phi$ on a shift $X_{\mathcal{H}}$, we want to extend $\phi$ to a transient potential $\widetilde{\phi}$ on $X_{\mathcal{G}}$. Our strategy is to "build" $\mathcal{G}$ from $\mathcal{H}$ by adding the edges and vertices of $\mathcal{G}$ to $\mathcal{H}$ one at a time and, at each step, extend $\phi$ to this new intermediate shift. Then we check that the common extension (to $X_{\mathcal{G}}$) is transient.

### 4.1. **Technical Lemmas.**

To carry out this construction, we will need two tools which are described in lemmas 4.1 and 4.3 and deal with the cases of adding a new edge and a new vertex to $\mathcal{H}$, respectively.

**Lemma 4.1.** *Let $\phi$ be a transient Markov potential on a shift $X_{\mathcal{H}}$. Let $\widetilde{\mathcal{H}}$ be the graph $\mathcal{H}$ with one additional edge added. Then $\phi$ can be extended to a transient Markov potential $\widetilde{\phi}$ on $X_{\widetilde{\mathcal{H}}}$ with the same pressure as $\phi$.*

*Proof.* Suppose the new edge is $a \to b$ and let

$$p : a \xrightarrow{L(p)} b$$

be a path in $\mathcal{H}$ from $a$ to $b$ (possible by topological mixing). We will define a new potential $\widetilde{\phi}$ on $X_{\widetilde{\mathcal{H}}}$ by setting $\widetilde{\phi}(x_0, x_1) = \phi(x_0, x_1)$ whenever $x_0 \to x_1$ is an allowed transition in $\mathcal{H}$ and $\widetilde{\phi}(a, b) = -N$ (to be specified later). We claim that for a sufficiently large value of $N$, the potential $\widetilde{\phi}$ will be transient and have $P_G(\widetilde{\phi}) = P_G(\phi)$.

Note that $Z_n(\widetilde{\phi}, a) \geq Z_n(\phi, a)$ since the former sum contains the latter as well as other entries. So $P_G(\widetilde{\phi}) \geq P_G(\phi)$. Now observe that we can write $Z_n^*(\widetilde{\phi}, a)$ in the following way (taking $k := L(p)$):

$$\begin{aligned} Z_n^*(\widetilde{\phi}, a) &= \sum_{T^n x = x} e^{\widetilde{\phi}_n(x)} 1_{[\varphi_a = n]}(x) \\ &= \sum_{T^n x = x, x \in X_{\mathcal{H}}} e^{\phi_n(x)} 1_{[\varphi_a = n]}(x) + \sum_{T^n x = x, x \notin X_{\mathcal{H}}} e^{-N} e^{\phi_{n-1}(Tx)} 1_{[\varphi_a = n]}(x) \\ &\leq Z_n^*(\phi, a) + e^{-N} e^{-\phi_{k-1}(\overline{x})} Z_{n+k-2}^*(\phi, a) \end{aligned}$$

where $\overline{x}$ is a fixed (but arbitrary) element of $[(p)_0, \ldots, (p)_{L(p)-1}] \cap X_{\mathcal{H}}$. Therefore,

$$r_\phi(x) \leq r_{\widetilde{\phi}}(x) \leq r_\phi(x) + \frac{C(N)}{\varepsilon^{k-2}} r_\phi(x) = \Big(1 + \frac{C(N)}{\varepsilon^{k-2}}\Big) r_\phi(x) \tag{4.1}$$

for any $x \in [\varepsilon, R]$, where $\varepsilon > 0$ is arbitrary, $R$ is the radius of convergence of $r_\phi(x)$ and $C(N) = e^{-N} e^{-\phi_{k-1}(\overline{x})}$ can be made arbitrarily small by taking $N$ sufficiently large. Therefore $r_\phi(x)$ and $r_{\widetilde{\phi}}(x)$ have the same radius of convergence: $\lambda^{-1} = e^{-P_G(\phi)}$ (by transience of $\phi$ and equation (1.1), this is the radius of convergence of $r_\phi(x)$). But, by transience of $\phi$, $t_\phi(x) = \frac{1}{1-r_\phi(x)}$ and the function converges at $\lambda^{-1}$ so $r_\phi(\lambda^{-1}) < 1$, and so for sufficiently large $N$ we have $r_{\widetilde{\phi}}(\lambda^{-1}) < 1$ and hence $t_{\widetilde{\phi}}(\lambda^{-1}) < \infty$. This gives us $P_G(\widetilde{\phi}) \leq P_G(\phi)$ and our previous estimate shows that $P_G(\widetilde{\phi}) = P_G(\phi)$. Since $t_{\widetilde{\phi}}(e^{-P_G(\widetilde{\phi})}) < \infty$, $(X_{\widetilde{\mathcal{H}}}, \widetilde{\phi})$ is transient. □

**Lemma 4.2.** *Let $\phi$ be a transient Markov potential on a shift $X_{\mathcal{H}}$. Let $\widetilde{\mathcal{H}}$ be the graph $\mathcal{H}$ together with a finite collection $v_1, \ldots, v_r$ of new vertices connected by a simple path*

$$a \to v_1 \to v_2 \to \cdots \to v_r \to b$$

*for some $a, b \in \mathcal{S}_{\mathcal{H}}$. Then $\phi$ can be extended to a transient Markov potential $\widetilde{\phi}$ on $X_{\widetilde{\mathcal{H}}}$ with $P_G(\widetilde{\phi}) = P_G(\phi)$.*

*Proof.* The proof is the same as in Lemma 4.1 except we set $\widetilde{\phi}(a, v_1) = -N$ and $\widetilde{\phi}(v_i, v_{i+1}) = \widetilde{\phi}(v_r, b) = 0$. □

**Remark 1.** We actually get a stronger statement in the previous lemmas. Given any $\varepsilon > 0$ we can take $N$ so large that $r_{\widetilde{\phi}}(e^{-P_G(\widetilde{\phi})}) \leq r_\phi(e^{-P_G(\phi)}) + \varepsilon$ (this follows from (4.1)).

**Lemma 4.3.** *Let $\mathcal{H}$ be a topologically mixing subgraph of $\mathcal{G}$. If $\phi$ is a transient Markov potential on $X_{\mathcal{H}}$ and $v \in \mathcal{S}_{\mathcal{G}} \setminus \mathcal{S}_{\mathcal{H}}$, then there is a topologically mixing graph $\mathcal{K}$ with*

$$\mathcal{H} \subseteq \mathcal{K} \subseteq \mathcal{G}$$

*such that $v \in \mathcal{K}$ and $\phi$ can be extended to a transient Markov potential on $X_{\mathcal{K}}$ with pressure $P_G(\phi)$.*

*Proof.* Given a vertex $v \in \mathcal{S}_{\mathcal{G}} \setminus \mathcal{S}_{\mathcal{H}}$ there are vertices $a, b \in \mathcal{S}_{\mathcal{H}}$ so that there are simple paths (in $\mathcal{G}$) $a \xrightarrow{n} v$ and $v \xrightarrow{m} b$. Together these determine a path, $p : a \xrightarrow{n+m-1} b$ in $\mathcal{G}$ given by

$$p : a \to x_1 \to \cdots \to x_{n-2} \to v \to y_1 \to \cdots \to y_{m-2} \to b$$

and w.l.o.g. we can assume that $x_i, y_j \notin \mathcal{S}_{\mathcal{H}}$ for any $i, j$ (otherwise rechoose $a = x_i$ and $b = y_j$ appropriately so that we do have this property).

If $p$ is simple, then we can take $\mathcal{K} = \mathcal{H} \cup \{p\}$ and apply lemma 4.2 to extend $\phi$ to a transient Markov potential on $X_{\mathcal{K}}$ with pressure $P_G(\phi)$. Otherwise there is a repeated vertex (apart from possibly $a = b$) so there are $i, j$ with $1 \leq i \leq n-2$ and $1 \leq j \leq m-2$ such that $x_i = y_j$. Let $i_1$ be the smallest possible index for which there exists $j_1$ with $x_{i_1} = y_{j_1}$. Then the path

$$p_1 : a \to x_1 \to \cdots \to x_{i_1} \to y_{j_1+1} \to \cdots \to b$$

has no repeated vertices (by minimality of $i_1$) and we can apply lemma 4.2 to extend $\phi$ to a transient Markov potential $\phi[1]$ on $X_{\mathcal{H}_1}$ with $P_G(\phi[1]) = P_G(\phi)$, where $\mathcal{H}_1 = \mathcal{H} \cup \{p_1\}$. Note that $i_2 := \max\{i \mid x_i \in \mathcal{S}_{\mathcal{H}_1}\} > \max\{i \mid x_i \in \mathcal{S}_{\mathcal{H}}\}$ (where $a = x_0$) and similarly $j_2 := \min\{j \mid y_j \in \mathcal{S}_{\mathcal{H}_1}\} < \min\{j \mid y_j \in \mathcal{S}_{\mathcal{H}}\}$. Take $a_1 = x_{i_2}$, $b_1 = y_{j_2}$ and consider the path

$$\widetilde{p} : a_1 \to x_{i_2+1} \to \cdots \to x_{n-2} \to v \to y_1 \to \cdots \to y_{j_2-1} \to b_1$$

which has $L(\widetilde{p}) < L(p)$. Repeat the above process, using $\widetilde{p}$ instead of $p$, to produce a graph $\mathcal{H}_2$ with $\mathcal{H}_1 \subset \mathcal{H}_2$ to which we can further extend $\phi[1]$ via lemma 4.2. Continue inductively until we have a sequence of graphs $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{H}_r$ with $v \in \mathcal{H}_r$ and a transient Markov potential $\phi[r]$ on $X_{\mathcal{H}_r}$ extending $\phi$ with $P_G(\phi[r]) = P_G(\phi)$. Taking $\mathcal{K} = \mathcal{H}_r$ gives the desired result. □

**Remark 2.** As in the previous remark, given any $\varepsilon > 0$, we arrange for $r_{\widetilde{\phi}}(\lambda^{-1}) = r_\phi(\lambda^{-1}) + \varepsilon$ (since we only extend $\phi$ through finitely many graphs to get to $\widetilde{\phi}$).

4.2. **Proof of Theorem 2.4.** Enumerate the vertices in the graph $\mathcal{G}$ as

$$\mathcal{S}_{\mathcal{G}} = \{v_1, v_2, v_3, \dots\}$$

In order to build the potential $\widetilde{\phi}$ on $X_{\mathcal{G}}$ we first produce a (possibly infinite) sequence of intermediate graphs

$$\mathcal{H} = \mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{I}$$
$$\mathcal{I} = \mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{G}$$

with the property that $\mathcal{I} = \cup \mathcal{H}_i$, $\mathcal{G} = \cup \mathcal{I}_i$ and

- $\mathcal{S}_{\mathcal{I}} = \mathcal{S}_{\mathcal{G}}$ (the graph $\mathcal{I}$ contains all vertices in the graph $\mathcal{G}$);
- $\mathcal{I}_{i+1}$ is the graph $\mathcal{I}_i$ with one additional edge added [as in lemma 4.1].

**Step 1**: *Producing the sequence of graphs $\mathcal{H}_i$.*

We produce the graphs $\mathcal{H}_i$ by induction. Begin by setting $\mathcal{H}_1 := \mathcal{H}$ and let $\phi_1 := \phi$. Apply lemma 4.3 to $\mathcal{H}_1 \subset \mathcal{G}$ with potential $\phi_1$ and vertex $v_1$. Let $\mathcal{H}_2$ and $\phi_2$ be the graph and potential (respectively) obtained from the lemma.

Now assume that we have constructed a sequence of graphs

$$\mathcal{H} = \mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{H}_r$$

and transient Markov potentials $\phi_i$ (on $X_{\mathcal{H}_i}$) so that $\phi_i$ is an extension of $\phi_{i-1}$ and $v_{i-1} \in \mathcal{H}_i$ for $2 \le i \le r$. Then apply lemma 4.3 to $\mathcal{H}_r \subset \mathcal{G}$ with potential $\phi_r$ and vertex $v_r$. Let $\mathcal{H}_{r+1}$ and $\phi_{r+1}$ be the graph and potential obtained from the lemma.

By induction, we get a sequence of graphs so that

$$\mathcal{I} := \bigcup_{i=1}^{\infty} \mathcal{H}_i$$

satisfies $\mathcal{S}_{\mathcal{I}} = \mathcal{S}_{\mathcal{G}}$.

**Step 2**: *Adding the vertices of $\mathcal{G}$ to $\mathcal{H}$.*

We will show that $\phi$ can be extended to a transient Markov potential $\varphi$ on $\mathcal{I}$ with $P_G(\varphi) = P_G(\phi)$.

Let $\phi_i$ be as defined in the previous step. Following the remark after lemma 4.3, we can arrange for $r_{\phi_i}(\lambda^{-1}) \le r_\phi(\lambda^{-1}) + \epsilon_0$ for all $i$. Let $\varphi$ be the potential on $\mathcal{I}$ defined by $\varphi(x_0, x_1) = \phi_i(x_0, x_1)$ where $i$ is any index for which $x_0 \to x_1$ is an allowed transition in $\mathcal{H}_i$. We claim that $P_G(\varphi) = P_G(\phi)$ and that $(X_{\mathcal{I}}, \varphi)$ is transient. This will complete step 1.

Suppose $Z_n^*(\varphi, a) < \infty \ \forall n$ but $\phi$ is recurrent. Set $\lambda = e^{P_G(\phi)}$. Since $P_G(\varphi) \geq P_G(\phi)$, there is some $R$ such that

$$\sum_{k=1}^{R} Z_k^*(\varphi, a)\lambda^{-k} > r_\phi(\lambda^{-1}) + 2\epsilon_0.$$

Find $i$ such that $\mathcal{H}_i$ has enough paths so that $Z_j^*(\phi_i, a) \geq Z_j^*(\varphi, a) - \frac{\epsilon_0 \lambda^j}{2^j} \ \forall 1 \leq j \leq R$ (possible since $Z_j^*(\varphi, a)$ is a sum that can be truncated with small error and the resulting sum is only over finitely many paths – we can find $i$ so that $\mathcal{H}_i$ contains all of these). Then $r_{\phi_i}(\lambda^{-1}) > r_\phi(\lambda^{-1}) + \epsilon_0$ which contradicts the definition of $\phi_i$. Therefore $r_\varphi(\lambda^{-1}) < r_\phi(\lambda^{-1}) + 2\epsilon_0 < 1$. Since $\varphi$ is Markov, this implies that $\varphi$ is transient on $X_{\mathcal{I}}$ and since $r_\varphi(\lambda^{-1}) < \infty$ we have $P_G(\varphi) = P_G(\phi)$. Thus, it remains only to show that $Z_n^*(\varphi, a) < \infty \ \forall n$.

Suppose $\exists n$ such that $Z_n^*(\varphi, a) = \infty$. We can find some finite set $F \subset \{x \mid T^n x = x, \varphi_a(x) = n\}$ such that

$$\sum_{x \in F} e^{\varphi_n(x)} > 2e^{nP_G(\phi)}.$$

Since $F$ is finite, we can find some $i$ such that $Z_n^*(\phi_i, a) > 2e^{nP_G(\phi)}$ ($F$ contains only finitely many paths). But $\phi_i$ is transient and Markov so $\sum_k Z_k^*(\phi_i, a)e^{-kP_G(\phi)} < 1$ (since $r_{\phi_i}(\lambda^{-1}) < 1$) and so $Z_n^*(\phi_i, a) < 2e^{nP_G(\phi)}$. Therefore $Z_n^*(\varphi, a) < \infty \ \forall n$.

Therefore, $\varphi$ is a transient Markov potential on $X_{\mathcal{I}}$ with $P_G(\varphi) = P_G(\phi)$.

**Step 3**: *Adding the edges of $\mathcal{G}$ to $\mathcal{I}$.*

We want to extend the potential $\varphi$ on $X_{\mathcal{I}}$ to a potential on $X_{\mathcal{G}}$. The argument is as above, using lemma 4.1 instead of 4.3. □

## 5. Shifts with Transient Potentials

In this section we define several classes of graphs and explicitly find transient potentials on their associated shifts. First we address a technical point.

Recall that the shift $X_{\mathcal{G}}$ is assumed to be topologically mixing. We will find a transient potential on $X_{\mathcal{G}}$ by finding a subgraph $\mathcal{H} \subseteq \mathcal{G}$ for which it is easy to define a transient Markov potential on $X_{\mathcal{H}}$ and use the extension theorem to extend to a transient Markov potential on $X_{\mathcal{G}}$. The graph $\mathcal{H}$ that we construct will often be only path connected (so $X_{\mathcal{H}}$ is topologically transitive). To force $X_{\mathcal{H}}$ to be topologically mixing we will assume the existence of an extra edge $a \to a$ for some $a \in \mathcal{S}_{\mathcal{H}}$ even if this edge does not appear in $\mathcal{G}$. The following lemma shows why this procedure does not cause a problem.

**Lemma 5.1.** *Let $\mathcal{G}$ be a directed graph and $X_{\mathcal{G}}$ topologically mixing. Let $a \in \mathcal{S}_{\mathcal{G}}$ and define $\widetilde{\mathcal{G}}$ to be the graph $\mathcal{G}$ with the additional edge $a \to a$ (if it was not already present). Then:*

1. *any transient Markov potential $\phi$ on $X_{\mathcal{G}}$ extends to a transient Markov potential $\varphi$ on $X_{\widetilde{\mathcal{G}}}$,*
2. *any transient Markov potential $\varphi$ on $X_{\widetilde{\mathcal{G}}}$ restricts to a transient Markov potential $\phi$ on $X_{\mathcal{G}}$.*

Thus if we construct a subgraph $\mathcal{H} \subseteq \mathcal{G}$ and $X_{\mathcal{H}}$ is topologically transitive, we can consider the subgraph $\widetilde{\mathcal{H}} \subseteq \widetilde{\mathcal{G}}$ obtained by adding the edge $a \to a$ to both $\mathcal{G}$ and $\mathcal{H}$. Now if we construct a transient Markov potential on $X_{\widetilde{\mathcal{H}}}$, it can be extended to a transient Markov potential on $X_{\widetilde{\mathcal{G}}}$ by theorem 2.4 and restricted to $X_{\mathcal{G}}$ to get a transient potential on $X_{\mathcal{G}}$. The upshot is that even if we need to assume the existence of an edge in $\mathcal{H}$ that isn't present in $\mathcal{G}$, any transient potential obtained on this graph gives rise to a transient potential on $X_{\mathcal{G}}$.

*Proof of Lemma* 5.1. For the first statement, assume that the edge $a \to a$ was not already present in $\mathcal{G}$ (otherwise no "extension" is necessary and we are done). Let $\phi$ be a transient Markov potential on $X_{\mathcal{G}}$. Define $\varphi$ on $X_{\widetilde{\mathcal{G}}}$ by $\varphi(x_0, x_1) := \phi(x_0, x_1)$ whenever $x_0 \to x_1$ in $\mathcal{G}$ and $\varphi(a, a) = -M$ (to be specified later). Then defining $r_\phi(x)$ and $r_\varphi(x)$ in terms of $Z_n^*(\phi, a)$ and $Z_n^*(\varphi, a)$ respectively, we get

$$r_\varphi(x) = e^{-M}x + r_\phi(x)$$

(in particular both series have the same radius of convergence). Since $\phi$ and $\varphi$ are Markov, we get

$$t_\varphi(x) = \frac{1}{1 - r_\varphi(x)} = \frac{1}{1 - e^{-M}x - r_\phi(x)}.$$

Now $t_\phi(x) = \frac{1}{1 - r_\phi(x)}$ converges at its radius of convergence, so for sufficiently large $M$ we get that $t_\varphi(x)$ converges at its radius of convergence too. So $\varphi$ is transient on $X_{\widetilde{\mathcal{G}}}$.

The second statement is similar; if $\phi$ is the restriction of $\varphi$ to $X_{\mathcal{G}}$ then $r_\varphi(x)$ and $r_\phi(x)$ have the same radius of convergence (since we only have deleted one path from the series defining it) and $r_\phi(x) < r_\varphi(x)$ for any $x$ for which both series converge. Thus $t_\phi(x) < t_\varphi(x)$ at their (common) radius of convergence, so $\phi$ is transient on $X_{\mathcal{G}}$. □

5.1. **Infinite Rays.** In this section we prove lemma 5.2 which shows that if $\mathcal{G}$ contains an infinite ray, then $X_{\mathcal{G}}$ has a transient potential.

**Definition 5.1.** *An* infinite forward ray *in a directed graph $\mathcal{G}$ is a path*

$$v_0 \to v_1 \to v_2 \to v_3 \to \cdots$$

*where $v_i \neq v_j$ for any $i \neq j$. Similarly an* infinite backward ray *in $\mathcal{G}$ is a path*

$$w_0 \leftarrow w_1 \leftarrow w_2 \leftarrow w_3 \leftarrow \cdots$$

*where $w_i \neq w_j$ for any $i \neq j$.*

**Lemma 5.2.** *If $\mathcal{G}$ contains an infinite forward or backward ray, then $X_{\mathcal{G}}$ has a transient potential.*

*Proof.* Suppose $\mathcal{G}$ has an infinite forward ray

$$b_0 \to b_1 \to b_2 \to b_3 \to b_4 \to b_5 \to \cdots$$

We make two definitions:

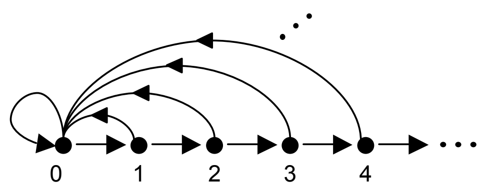
$$\begin{aligned} R &:= \{b_i \mid i \in \mathbb{N}_0\} \\ R_{i,j} &:= \{b_i, b_{i+1}, \ldots, b_{j-1}, b_j\} \text{ for any } i < j. \end{aligned}$$

The strategy for the proof is as follows. There are two particular graphs whose shifts have transient potentials: the *backwards renewal shift* and the *natural numbers* (defined below). Both contain infinite forward rays and turn out to be prototypes for all graphs containing such rays in the sense that any graph $\mathcal{G}$ (which determines a topologically mixing shift) has a subgraph with enough similarities to one of the prototypes that we can guess a transient potential on it by analogy with the transient potential on the prototype. We show how to find this subgraph and define the transient potential.

**Case 1 (Backwards Renewal Shift)**: Suppose there is some $k_0 \in \mathbb{N}_0$ such that for infinitely many $l > k_0$ (say for $l_1, l_2, \dots$) there is a path (w.l.o.g. a simple path)

$$p_i : b_{l_i} \xrightarrow{L(p_i)} b_{k_0}$$

such that $(p_i)_j \notin R$ for any $0 < j < L(p_i) - 1$; that is there is a path from $b_{l_i}$ to $b_{k_0}$ which is disjoint from the ray except at its endpoints. We claim that $\mathcal{G}$ contains a subgraph which is analogous to the *backwards renewal shift* (this directed graph has $\mathcal{S} = \mathbb{N}_0$, a directed edge from $n \to (n+1)$ for every $n \geq 0$ and a directed edge $n \to 0$ for every $n \geq 0$ – see figure below).




The Backwards Renewal Shift

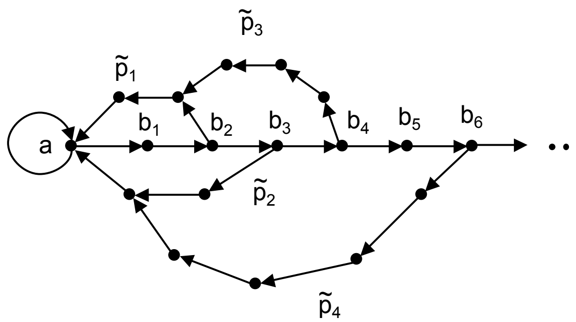
Without loss of generality we may assume that $k_0 = 0$ (otherwise consider the sub-ray starting from $b_{k_0}$) and, for notational convenience, we set $a := b_0$. Moreover let $V(p_1, p_2, \dots, p_k)$ be the set of vertices in the paths $p_1, \dots, p_k$.

The paths $\{p_i\}_{i=0}^{\infty}$ may intersect each other in complicated ways. We will modify the paths so that whenever two paths meet, they coalesce. Set $\widetilde{p}_1 := p_1$. Now $(\widetilde{p}_1)_{L(\widetilde{p}_1)-1} = a = (p_2)_{L(\widetilde{p}_2)-1}$, so let $i_0$ be the minimal index so that $(p_2)_{i_0} \in V(\widetilde{p}_1)$ and define $j_0$ so that $(p_2)_{i_0} = (\widetilde{p}_1)_{j_0}$. We define a new path $\widetilde{p}_2$ from $b_{l_2}$ to $a$ by following $p_2$ until the first point of intersection with $\widetilde{p}_1$ and then following $\widetilde{p}_1$ the rest of the way to $a$; specifically we set

$$(\widetilde{p}_2)_i = \begin{cases} (p_2)_i \text{ for } 0 \leq i \leq i_0 \\ (\widetilde{p}_1)_{j_0 - i_0 + i} \text{ for } i_0 < i \leq L(\widetilde{p}_1) + i_0 - i_0 - 1. \end{cases}$$

Recursively define $\widetilde{p}_{k+1}$ from $\widetilde{p}_1, \dots, \widetilde{p}_k$ by setting $\widetilde{p}_{k+1}$ to be the path $p_{k+1}$ until the first point of intersection with $V(\widetilde{p}_1, \dots, \widetilde{p}_k)$ and then following $\widetilde{p}_i$ the rest of the way to $a$ (where $i$ is the index of one of the paths met at this point). Note that the graph obtained by taking the union of all the paths $\widetilde{p}_i$ is a subgraph of $\mathcal{G}$ in which every vertex has outgoing degree one (that is from any vertex there is a unique successor vertex but possibly many predecessor vertices).

Finally let $\mathcal{H}$ be the subgraph of $\mathcal{G}$ obtained by taking all vertices and edges contained in the ray $a \to b_1 \to b_2 \to \cdots$ and all vertices and edges contained in the paths $\widetilde{p}_i$ for $i = 1, 2, \dots$ (as well as the edge $a \to a$ using lemma 5.1).




The subgraph $\mathcal{H}$

By construction, for each $i$ there is a unique first return loop (in $\mathcal{H}$) based at $a$ that passes through the first edge of $\widetilde{p}_i$ and, moreover, all first return loops can be described in this way (since they must contain at least one edge not contained in $R$ and so must pass through the first edge of some $\widetilde{p}_i$). Define a potential

$$\phi(x_0, x_1) = \begin{cases} \log \frac{1}{3k^2} & \text{if } x_0 = b_{l_k},\ x_1 = (\widetilde{p}_k)_1 \\ 0 & \text{otherwise.} \end{cases}$$

We claim that the series $r_\phi(x)$ converges at $x = 1$ and diverges for $x > 1$. First,

$$r_\phi(1) = \sum_{n=0}^{\infty} Z_n^*(\phi, a) = \sum_{k=1}^{\infty} \frac{1}{3k^2} < 1.$$

On the other hand, the length of the (unique) first return loop that passes through the edge $b_{l_k} \to (p_k)_1$ has length at least $k$, so for any $x > 1$ we have

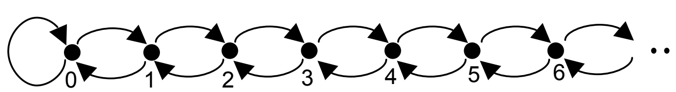
$$r_\phi(x) = \sum_{n=0}^{\infty} Z_n^*(\phi, a) x^n \geq \sum_{n=0}^{\infty} \frac{x^n}{3n^2} = \infty.$$

Thus $r_\phi(x)$ converges at its radius of convergence, and $r_\phi(1) < 1$ so $\phi$ is transient by equation (1.1).

**Case 2 (The Natural Numbers)**: If case 1 doesn't hold, then for every $k$ there are at most *finitely* many $l > k$ (say $l_1^k < l_2^k < \cdots < l_{r_k}^k$) such that there is a path

$$p_l : b_l \xrightarrow{L(p_l)} R_{0,k}$$

with $(p_l)_j \notin R$ for $0 < j < L(p_l) - 1$. Now we claim that $\mathcal{G}$ contains a subgraph which is analogous to the nearest neighbor graph of the *natural numbers* (we will denote this graph $\mathcal{G}(\mathbb{N}_0)$ – see figure below).




$\mathcal{G}(\mathbb{N}_0)$

Set $m_k := l_{r_k}^k$ (the maximal index with the above property). Let

$$p_0 : b_{m_0} \xrightarrow{L(p_1)} a$$

be a simple path with $(p_1)_i \notin R$ for any $0 < i < L(p_1) - 1$. Define a sequence $\{M_i\}$ by $M_0 := m_0$ and $M_{i+1} = m_{M_i}$. By construction, for every $k \geq 1$, there is a simple path

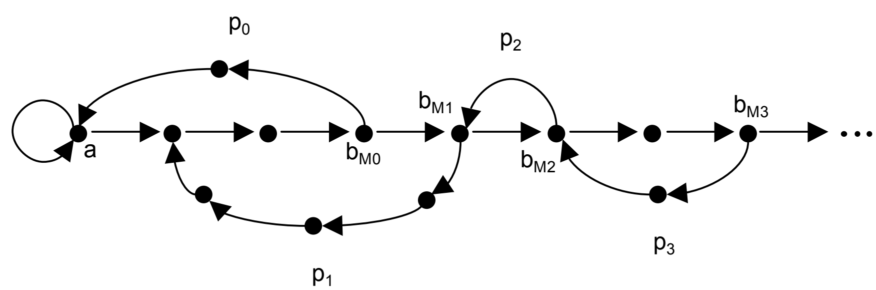
$$p_k : b_{M_k} \xrightarrow{L(p_k)} R_{0,M_{k-1}}$$

such that $(p_k)_i \notin R$ for any $0 < i < L(p) - 1$. Moreover, by maximality of $M_{k-1}$, we have

$$(p_k)_{L(p_k)-1} \in R_{1+M_{k-2},M_{k-1}}.$$

Finally note that $V(p_i) \cap V(p_j) = \emptyset$ if $|i - j| \neq 1$ and either $V(p_{i+1}) \cap V(p_i) = \emptyset$ or $V(p_{i+1}) \cap V(p_i) = (p_i)_0 = (p_{i+1})_{L(p_{i+1})-1}$, since otherwise (if $p_i$ and $p_j$ meet at some vertex not in $R$) we could follow $p_i$ to this meeting point and then follow $p_j$ back to $R$ to obtain a contradiction of the maximality of $M_j$.

Let $\mathcal{H}$ be the subgraph of $\mathcal{G}$ obtained by taking all vertices and edges in the ray and the paths $\{p_i\}_{i=0}^{\infty}$ (and $a \to a$ by lemma 5.1).




The subgraph $\mathcal{H}$

All vertices in $\mathcal{H}$ except for those in the set $A := \{b_{M_0}, b_{M_1}, b_{M_2}, \dots\}$ have outgoing degree one and each vertex in $A$ has outgoing degree two (namely, the vertices $b_{M_k+1}$ and $(p_k)_1$ are the successors of $b_{M_k}$). Thus a path in $\mathcal{H}$ is completely determined by a description of which successor is chosen each time a vertex in $A$ is crossed (note the similarity with the natural numbers where a path is just a description of whether to go right or left at any particular vertex).

We now work to obtain a precise description of the first return loops in $\mathcal{H}$ based at $a$. Define a map $\alpha : R \to \mathbb{N}_0$ by

$$\alpha(b_i) = \begin{cases} 0 & \text{if } i = 0 \\ j & \text{if } b_i \in R_{M_{j-1}+1,M_j}. \end{cases}$$

Further, define a function $\beta : X_{\mathcal{H}} \to \mathbb{N}_0^{\mathbb{N}_0}$ by

$$\beta(x)_j = \alpha(x_{k_j})$$

where $k_0, k_1, \dots$ are the indices for which $x_i \in A$. Note that $\beta$ gives a bijection between paths in $\mathcal{H}$ starting at $a$ and paths in $\mathcal{G}(\mathbb{N}_0)$ starting at 0. Moreover, a path in $\mathcal{H}$ is a loop (based at $a$) if and only if its image under $\beta$ is a loop in $\mathbb{N}$ (based at 0).

Now the potential $\phi(x_0, x_1) \equiv -\log(2)$ has finite discriminant on $X_{\mathcal{G}(\mathbb{N}_0)}$ and $P_G(\phi) = 0$ (see proof in Appendix B). We define an analogous potential on $X_{\mathcal{H}}$:

$$\widetilde{\phi}(x_0, x_1) \equiv -\log(2) \cdot 1_A(x_0)$$

Observe that if $T^n y = y$ and $y_0 \in [a]$ is a loop in $\mathcal{H}$ based at $a$, then

$$e^{\widetilde{\phi}_n(y)} = e^{\phi_{m_y}(\beta(y))}$$

where $m_y$ is the length of the loop $\beta(y)$ in $\mathbb{N}$. Moreover the map $\beta$ cannot increase the length of a loop (but can decrease it). Therefore

$$r_{\widetilde{\phi}}(1) = r_\phi(1) < \infty$$
$$r_{\widetilde{\phi}}(x) \geq r_\phi(x) = \infty \text{ for } x > 1.$$

Therefore $\widetilde{\phi}$ (on $X_{\mathcal{H}}$) has finite discriminant and so, as in the previous case, there is a transient potential on $X_{\mathcal{G}}$.

The case with a backward infinite ray is similar. □

5.2. **Double Bouquets.** We now consider graphs without infinite rays. The main result in this section is lemma 5.4 which shows that if $\mathcal{G}$ has infinitely many disjoint simple loops, it has a transient potential. This is shown with the use of a special kind of subgraph called a *double bouquet* defined below.

**Definition 5.2.** *We say a graph $\mathcal{G}$ is a* double bouquet *if there is some vertex $a \in \mathcal{S}_{\mathcal{G}}$ such that $\mathcal{G}$ has the following form:*

- *Primary Loops: There is a set of countably many simple first return loops based at $a$*

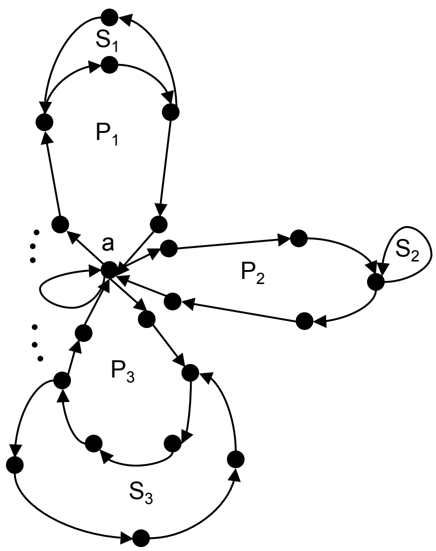
  $$\{P_1, P_2, \dots\}.$$

  *Moreover, there is a constant $C_0$ so that the loops are disjoint apart from their starting point and their last $C_0$ vertices. Specifically:*

  $$(P_i)_{L(P_i)-k-1} = (P_j)_{L(P_j)-k-1} \text{ for every } 0 \leq k \leq C_0$$

  $$(P_i)_k \notin V(P_j) \text{ for any } 0 < k < L(P_i) - C_0 - 1.$$

- *Secondary Loops: For each $i$ there exists $k_i \notin \{0, L(P_i) - 1\}$ and a simple loop $S_i$ based at $(P_i)_{k_i}$ whose vertex set is disjoint from $V(P_j) \cup V(S_j)$ for $j \neq i$. Moreover $a \notin V(S_i)$ and in the graph that consists of all edges and vertices in $P_i$ and $S_i$, there is exactly one vertex with outgoing degree two.*
- *There are no other vertices or edges in $\mathcal{G}$ except possibly the edge $a \to a$.*




A Double Bouquet (with $C_0 = 0$)

**Lemma 5.3.** *If $\mathcal{G}$ is a double bouquet and $X_{\mathcal{G}}$ is topologically mixing, then $X_{\mathcal{G}}$ has a transient Markov potential.*

*Proof.* Define $b_k := (P_k)_1$ so that the edge $a \to b_k$ is not an edge in $P_i$ for any $i \neq k$ (and is not an edge in any $S_i$). Next let $c_k, d_k \in V(S_k)$ be vertices such that $c_k \to d_k$ is an edge in $S_k$ but *not* in $P_k$. Define a potential $\phi$ by

$$\phi(x) = \phi(x_0, x_1) := \begin{cases} \log(\frac{k}{k+1}) & \text{if } x_0 = c_k,\, x_1 = d_k \\ \log(\frac{1}{k^3}) & \text{if } x_0 = a,\, x_1 = b_k \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\ell$ is a first return loop (of length greater than one) in $X_{\mathcal{G}}$ based at $a$, then $(\ell)_1 = b_i$ for some $i$. Moreover, by definition of the loop $S_i$, there is some number $n$ so that $\ell$ is the loop running around $P_i$ once and $S_i$ $n$ times (it cannot contain any other $P_j$ or $S_j$ as a subloop since it is a first return loop). We will show that that $r_\phi^a(x)$ converges for $x = 1$ and diverges for $x > 1$ (and hence has finite discriminant and finite pressure).

Recall that $r_\phi(1) = \sum_{n=1}^{\infty} Z_n^*(\phi, a)$. The contribution to the sum from the $k^{th}$ component of $\mathcal{G}$ (namely $P_k \cup S_k$) is

$$\sum_{m=0}^{\infty} e^{\phi(a,b_k)} e^{m\phi(c_k,d_k)}.$$

So the sum in question is

$$\begin{aligned} r_\phi(1) &= 1 + \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k^3} \Big(\frac{k}{k+1}\Big)^m \\ &= 1 + \sum_{k=1}^{\infty} \frac{1}{k^3} \sum_{m=0}^{\infty} \Big(\frac{k}{k+1}\Big)^m \\ &= 1 + \zeta(2) + \zeta(3). \end{aligned}$$

(here the 1 appears because of the edge from $a$ to itself).

Finally, we show that $r^a_\phi(x)$ diverges for $x>1$. Note that $L(P_k), L(S_k) \geq 1$.

$$\begin{aligned} r_\phi(x) &= 1 + \sum_{k=1}^{\infty} \frac{1}{k^3} x^{L(P_k)} + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{k^3}\Big(\frac{k}{k+1}\cdot x^{L(S_k)}\Big)^m x^{L(P_k)} \\ &\geq \sum_{k=1}^{\infty} \frac{1}{k^3} x^{L(P_k)} \sum_{m=1}^{\infty}\Big(\frac{k}{k+1}\cdot x\Big)^m, \text{ when } x>1 \\ &\geq \frac{1}{k_0^3}\sum_{m=1}^{\infty}\Big(\frac{k_0}{k_0+1}\cdot x\Big)^m = \infty \end{aligned}$$

where $k_0$ is some integer with $\big(\frac{k_0}{k_0+1}\cdot x\big) > 1$.

So the series $r_\phi(x)$ converges at its radius of convergence and so $\phi$ has finite discriminant and therefore a transient potential by lemma 1.1. □

**Lemma 5.4.** *If $\mathcal{G}$ has infinitely many pairwise disjoint simple loops (based at different points), then $X_\mathcal{G}$ has a transient Markov potential.*

*Proof.* We will assume that $\mathcal{G}$ does not contain an infinite ray (otherwise we are done by lemma 5.2) and show that $\mathcal{G}$ contains a double bouquet (and so a transient potential by lemma 5.3). The proof involves a number of inductive constructions, so we first give a brief overview of the steps.

*Overview of Steps.* We attempt to build a double bouquet. The idea is to view the infinite collection of disjoint loops (call them $\ell_1, \ell_2, \dots$) as the "secondary" loops and to build the primary loops.

Recall the properties of the primary loops $\{P_1, P_2, \dots\}$:

(1) The loops are simple and are all based at a common vertex $a$;
(2) The loop $P_i$ intersects $S_i$ so that any first return loop in $P_i \cup S_i$ based at $a$ is obtained by going around $P_i$ once and $S_i$ some number of times;
(3) For any $i, j$ the final $C_0$ vertices of $P_i$ and $P_j$ coincide;
(4) For any $i \neq j$, $P_i$ and $P_j$ are disjoint apart from their starting point and their final $C_0$ vertices.

*Step 1*: We fix $a \in \mathcal{S}$ and construct a path $\widetilde{p}_i$ from $a$ to $\ell_i$ for each $i$. We ensure that $V(\widetilde{p}_i) \cap V(\widetilde{p}_j) = \{a\}$ for $i \neq j$ and that $\widetilde{p}_i$ does not intersect $\ell_j$ for any $j \neq i$.

*Step 2*: We construct a path $\widetilde{\rho}_i$ from $\ell_i$ to $a$ for each $i$. We ensure that for some constant $C_0$, $V(\widetilde{\rho}_i) \cap V(\widetilde{\rho}_j)$ is exactly the final $C_0$ vertices of $\widetilde{\rho}_i$ for each $i \neq j$.

We would now *like* to let $P_i$ be the loop obtained by following $\widetilde{p}_i$ from $a$ to $\ell_i$, then following $\ell_i$ from the end of $\widetilde{p}_i$ to the beginning of $\widetilde{\rho}_i$, and finally following $\widetilde{\rho}_i$ back to $a$. Unfortunately these loops may not be of the proper form for two reasons: first the loops might not be disjoint outside of their starting point and their final $C_0$ vertices and, second, the loops might not be simple.

*Step 3*: We pass to a subsequence of these loops so that they are disjoint apart from their initial vertex and their final $C_0$ vertices.

*Step 4*: We modify the graph obtained from $\widetilde{p}_i$, $\widetilde{\rho}_i$ and $\ell_i$ so that the primary loop is simple (in some cases, this will require us to let the secondary loop be a different loop in this graph than $\ell_i$).

The result of these four steps will give us a double bouquet.

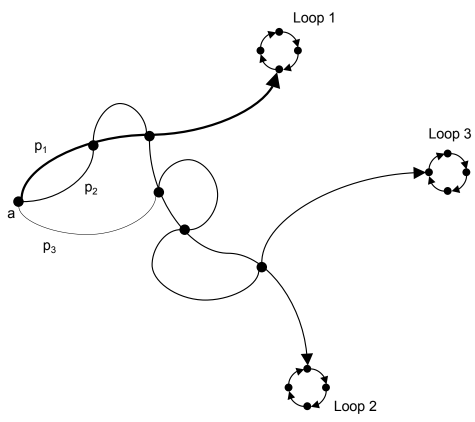
**Step 1**: The first step is to build a path from $a$ to each of the "secondary" loops and give these paths a simple graph structure.

Fix some $a \in \mathcal{S}_{\mathcal{G}}$ and suppose there are infinitely many disjoint loops in $X$; enumerate these as $\ell_1, \ell_2, \ldots$ (wlog assume that $a \notin \ell_i$ for any $i$). For each $i$, let

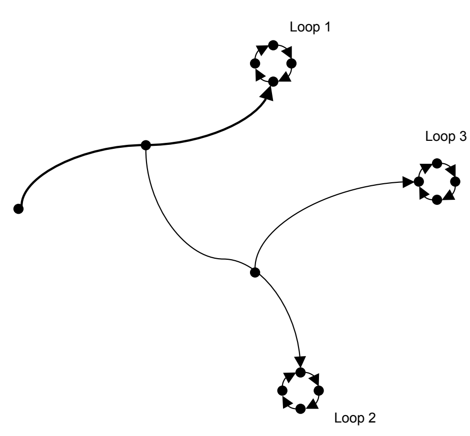
$$p_i : a \xrightarrow{L(p_i)} V(\ell_i)$$

be a simple path from $a$ to some vertex in $\ell_i$. Without loss of generality, we can assume that $V(p_i) \cap V(\ell_j) = \emptyset$ for any $i \neq j$ (otherwise $p_i$ intersects at most finitely many loops which can be deleted from our list; we inductively produce an infinite subsequence of loops so that the associated paths do not intersect more than one of them). The paths $\{p_i\}$ may intersect each other in complicated ways. We will define a new collection of paths $\{\widetilde{p}_i\}$ so that $\widetilde{p}_k$ connects $a$ to $V(\ell_k)$ and the union of $\{\widetilde{p}_i\}$ has a "branching" structure (the paths will be initially identical for a fixed number of vertices and then they will "branch off" from each other and be disjoint thereafter).

Begin by setting $\widetilde{p}_1 := p_1$. Now we proceed inductively. Suppose we have constructed simple paths $\widetilde{p}_1, \ldots, \widetilde{p}_k$ with $V(\widetilde{p}_1, \ldots, \widetilde{p}_k) \subseteq V(p_1, \ldots, p_k)$ so that the graph $V(\widetilde{p}_1) \cup \cdots \cup V(\widetilde{p}_k)$ has the structure of a tree, we construct $\widetilde{p}_{k+1}$. Since $(p_{k+1})_{L(p_{k+1}-1)} \notin V(\widetilde{p}_1, \ldots, \widetilde{p}_k)$ let $i_{k+1}$ be the maximal index such that $(p_{k+1})_{i_{k+1}} \in V(\widetilde{p}_1, \ldots, \widetilde{p}_k)$, say $(p_{k+1})_{i_{k+1}} = (\widetilde{p}_j)_n$ for some $j, n$. Define $\widetilde{p}_{k+1}$ to be the (simple) path that follows $\widetilde{p}_j$ until vertex $n$ and then follows $p_{k+1}$ the rest of the way to $\ell_{k+1}$.




The paths $p_i$




The paths $\widetilde{p}_i$

Note that for any $i \neq j$ the paths $\widetilde{p}_i$ and $\widetilde{p}_j$ are initially the same path and then split. Specifically, there is some index $s_{i,j}$ such that

$$\begin{aligned}
&(\widetilde{p}_i)_k = (\widetilde{p}_j)_k \text{ for all } 0 \leq k \leq s_{i,j} \\
&(\widetilde{p}_i)_k \notin V(\widetilde{p}_j) \text{ for any } k > s_{i,j} \\
&(\widetilde{p}_j)_k \notin V(\widetilde{p}_i) \text{ for any } k > s_{i,j}.
\end{aligned}$$

Our goal is now to find a subsequence of paths for which $s_{i,j}$ is independent of $i$ and $j$ (that is, there is a single vertex up to which all paths are the same and after which all paths are disjoint). To start, note that $s_{1,j} \leq L(\widetilde{p}_1)$ for all $j$, so let $s_1$ be the largest index such that $s_{1,j} = s_1$ for infinitely many $j \in \mathbb{N}$. Let $I_1$ be the set of indices $I_1 := \{j \mid s_{1,j} = s_1\}$. Suppose we have constructed index sets

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_r$$

such that $I_k$ is infinite for all $k = 1, \ldots, r$ and $s_{k,j} = \mathit{const}$ for all $j \in I_k$; set $s_k$ to be this constant. Since $I_r$ is infinite there is some $s_{r+1}$ so that $s_{r+1,j} = s_{r+1}$ for infinitely many $j \in I_r$ (note that $s_{r+1} \geq s_r$ since any two paths in $I_r$ have the same first $s_r$ vertices). Let $I_{r+1} \subseteq I_r$ be the set of indices for which this is true. By induction we obtain an infinite descending sequence of index sets

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots .$$

Let $k_1 < k_2 < k_3 < \cdots$ be a sequence of indices such that $k_i \in I_i$ for all $i$. Then for fixed $i$ and any $j > i$ we have $s_{k_i,k_j} = s_{k_i}$. By construction, if the sequence $s_{k_1}, s_{k_2}, \ldots$ is unbounded then $X$ contains an infinite forward ray; contradicting our assumption that it does not (we reduced to this case since otherwise we are done by lemma 5.2).

If the sequence $\{s_{k_i}\}$ is bounded, say $N$ is such that $s_{k_i} = s_{k_j}$ for all $i, j > N$, then we consider the subsequence of paths with index set

$$\mathcal{I} := \{k_{N+i} \mid i = 1, 2, \ldots\}$$

Then there is some constant $C_1$ such that for any $i, j \in \mathcal{I}$,

$$\begin{array}{l}(\widetilde{p}_i)_k = (\widetilde{p}_j)_k \text{ for all } 0 \leq k \leq C_1 \\ (\widetilde{p}_i)_k \notin V(\widetilde{p}_j) \text{ for any } k > C_1 \\ (\widetilde{p}_j)_k \notin V(\widetilde{p}_i) \text{ for any } k > C_1.\end{array}$$

Without loss of generality we can assume that $C_1 = 0$ (by redefining $a$ to be the vertex $(\widetilde{p}_i)_{C_1}$). This is the "branching" structure we had desired – the paths $\widetilde{p}_i$ connect $a$ to loops $\ell_i$ and are disjoint except at their initial vertex..

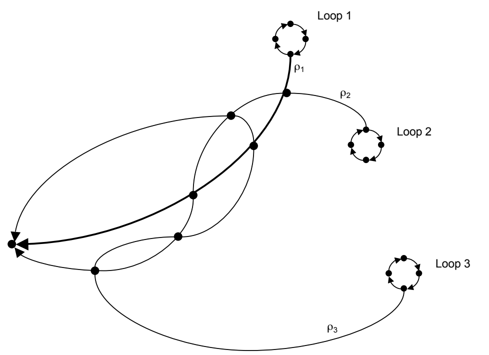
**Step 2**: We now build the rest of the "primary" loop by finding a path back from each "secondary" loop to $a$. In this step we will build these paths and give them a similar "branching" structure to those in the previous step.

For each $i \in \mathcal{I}$ (the index set from the previous step) let $\rho_i$ be a simple path from some vertex in $\ell_i$ to $a$:

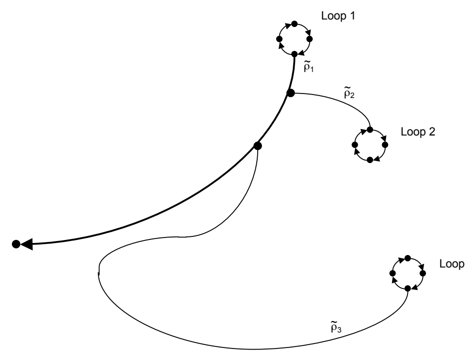
$$\rho_i : V(\ell_i) \xrightarrow{L(\rho_i)} a$$

such that $(\rho_i)_j \notin V(\ell_i)$ for any $j > 0$ (i.e. only the first vertex of $\rho_i$ is in $V(\ell_i)$). As before, we may assume that $V(\rho_i) \cap V(\ell_j) = \emptyset$ for every $i \neq j$ (otherwise $\rho_i$ intersects at most finitely many $\ell_j$ and we can inductively produce a subsequence where the condition holds). It may, again, be the case that the paths $\rho_{i_j}$ intersect each other in complicated ways. As before we will define a new collection of paths $\widetilde{\rho}_j$ so that $\widetilde{\rho}_j$ connects $\ell_j$ to $a$ and this time the paths $\{\widetilde{\rho}_i\}$ have a type of "reverse branching" structure (the paths are initially disjoint and all coalesce at a common vertex).

For convenience write $\mathcal{I} = \{i_1, i_2, \cdots\}$ with $i_1 < i_2 < \cdots$. Set $\widetilde{\rho}_{i_1} := \rho_{i_1}$. As before, we proceed by induction. Suppose we have constructed simple paths $\widetilde{\rho}_{i_1}, \ldots, \widetilde{\rho}_{i_k}$, we will show how to construct $\widetilde{\rho}_{i_{k+1}}$. Let $j_{k+1}$ be the *least* index so that $(\rho_{i_{k+1}})_{j_{k+1}} \in V(\widetilde{\rho}_{i_1}, \dots, \widetilde{\rho}_{i_k})$, say $(\rho_{i_{k+1}})_{i_{k+1}} = (\widetilde{\rho}_l)_m$ for some $l, m$. Define $\widetilde{\rho}_{i_{k+1}}$ to be the simple path that follows $\rho_{i_{k+1}}$ for the first $j_{k+1}$ vertices and then follows $\widetilde{\rho}_l$ thereafter.




The paths $\rho_i$




The paths $\widetilde{\rho}_i$

Then, as before, for any $i, j \in \mathcal{I}$ with $i \neq j$, there is some index $\widetilde{s}_{i,j}$ so that

$$\begin{array}{ll} (\widetilde{\rho}_i)_{L(\widetilde{\rho}_i)-k-1} = (\widetilde{\rho}_j)_{L(\widetilde{\rho}_j)-k-1} & \text{for all } 0 \le k \le \widetilde{s}_{i,j}; \\ (\widetilde{\rho}_i)_k \notin V(\widetilde{\rho}_j) & \text{for any } k < L(\widetilde{\rho}_i) - \widetilde{s}_{i,j} - 1; \\ (\widetilde{\rho}_j)_k \notin V(\widetilde{\rho}_i) & \text{for any } k < L(\widetilde{\rho}_j) - \widetilde{s}_{i,j} - 1. \end{array}$$

This means precisely that the paths are initially disjoint and at some point coalesce. Similar to before, either there is a backward infinite ray contained in $X$ (contradicting our assumption) or we can pass to a subsequence with index set $\mathcal{J} \subseteq \mathcal{I}$ so that the number $\widetilde{s}_{i,j}$ is independent of $i, j \in \mathcal{J}$. This gives exactly the "reverse branching" structure that we desired. Define $T$ to be the common set of vertices shared by the paths $\{\widetilde{\rho}_i \mid i \in \mathcal{J}\}$. Note that $T$ is the final $|T|$ vertices in each path $\widetilde{\rho}_i$ $(i \in \mathcal{J})$. Finally, we will be working only with the subsequence determined by $\mathcal{J}$ for the remainder of the proof, so for notational convenience we will assume that $\mathcal{J} = \mathbb{N}$ (by renaming our paths if necessary).

**Step 3**: As previously stated, we would like for the "primary" loops in our construction to be the loops determined by the paths $\widetilde{p}_i$ and $\widetilde{\rho}_i$ (and the vertices in $\ell_i$ needed to connect these two paths). There are two issues left to consider. First these "primary" loops may not be simple if $\widetilde{p}_i$ and $\widetilde{\rho}_i$ intersect in places other than their endpoints. We will resolve this in step four. The other issue is that a path $\widetilde{p}_i$ may intersect a number of different $\widetilde{\rho}_j$ (or vice versa) so that the "primary" loops are not disjoint outside of their final $|T|$ $(=: C_0)$ vertices. We resolve this issue here.

We will pass to a subsequence of triples $\{(\ell_i, \widetilde{p}_i, \widetilde{\rho}_i)\}_{i \in \mathbb{N}}$ so that $V(\widetilde{p}_i) \cap V(\widetilde{\rho}_j) = \{a\}$ for every $i \neq j$. We accomplish this inductively: take $c_1 := 1$ and observe that $\widetilde{\rho}_{c_1}$ intersects at most finitely many $\widetilde{p}_k$ outside of the set $\{a\}$ and similarly $\widetilde{p}_{c_1}$ intersects at most finitely many $\widetilde{\rho}_k$ outside of the set $T$ (defined in the previous step). Let

$$K_1 := \big\{k \in \mathbb{N} \mid V(\widetilde{p}_{c_1}) \cap V(\widetilde{\rho}_k) \subseteq T \text{ and } V(\widetilde{\rho}_{c_1}) \cap V(\widetilde{p}_k) = \{a\}\big\}.$$

We construct sets $K_i$ by induction. Assume that we have constructed a sequence of infinite index sets

$$K_1 \supseteq \cdots \supseteq K_r$$

and integers $c_1 < \cdots < c_r$ so that $c_i \in K_{i-1}$ (we consider $K_0 := \mathbb{N}$) so that whenever $i \in K_n$, we have $V(\widetilde{p}_{c_n}) \cap V(\widetilde{\rho}_i) \subseteq T$ and $V(\widetilde{\rho}_{c_n}) \cap V(\widetilde{p}_i) = \{a\}$. Let $c_{r+1} \in K_r$ be some element with $c_{r+1} > c_r$. Define

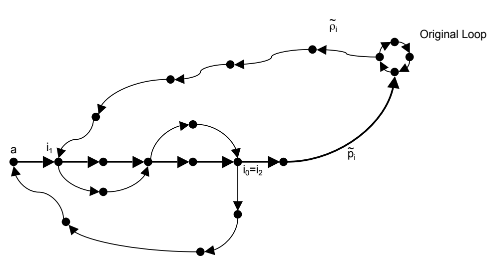
$$K_{r+1} := \big\{k \in K_r \mid V(\widetilde{p}_{c_{r+1}}) \cap V(\widetilde{\rho}_k) \subseteq T \text{ and } V(\widetilde{\rho}_{c_{r+1}}) \cap V(\widetilde{p}_k) = \{a\}\big\}.$$

This set is infinite since $\widetilde{p}_{r+1}$ intersects at most finitely many $\widetilde{\rho}_k$ outside of $T$ and $\rho_{r+1}$ intersects at most finitely many $p_k$ outside of $\{a\}$. Finally let $\mathcal{K} := \{c_1, c_2, \dots\}$. By construction, if $i, j \in \mathcal{K}$ and $i < j$ then $V(\widetilde{p}_i) \cap V(\widetilde{\rho}_j) \subseteq T$ and $V(\widetilde{\rho}_i) \cap V(\widetilde{p}_j) = \{a\}$. But there are only finitely many $j \in \mathcal{K}$ for which $V(\widetilde{p}_j) \cap (T \setminus \{a\}) \neq \emptyset$ (since the paths $\widetilde{p}_i$ are disjoint apart from their common starting point, so each vertex in $T \setminus \{a\}$ is contained in at most one $\widetilde{p}_i$). Let $\mathcal{K}' \subseteq \mathcal{K}$ be the (infinite) set of vertices for which $V(\widetilde{p}_i) \cap V(\widetilde{\rho}_j) = \{a\}$ for every $i, j \in \mathcal{K}'$ with $i \neq j$.

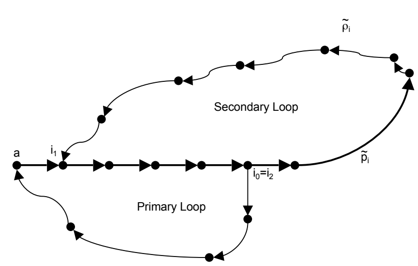
It is now the case that if $i, j \in \mathcal{K}'$ and $i \neq j$, then $V(\widetilde{p}_i) \cap V(\widetilde{\rho}_j) = \{a\}$, $V(\widetilde{\rho}_i) \cap V(\widetilde{\rho}_j) = T$ and $V(\ell_i) \cap V(\widetilde{\rho}_j) = \emptyset$. That is, if $\widetilde{P}_i$ is the loop formed by following $\widetilde{p}$ from $a$ to $\ell_i$, then following $\ell_i$ to from the last vertex in $\widetilde{p}_i$ to the first vertex in $\widetilde{\rho}_i$ and finally following $\widetilde{\rho}_i$ back to $a$, then the loops $\widetilde{P}_i$ are all based at a common vertex $a$ and are all disjoint apart from $\{a\}$ and their final $C_0 = |T|$ vertices.

**Step 4**: We finally must ensure that the paths $\widetilde{P}_i$ we have constructed are simple. Unfortunately this may not be the case – this situation is pictured below. We modify the triples $(\widetilde{p}_i, \widetilde{\rho}_i, \ell_i)$ so that the "primary" loops are simple.

Given a triple $(\ell_i, \widetilde{p}_i, \widetilde{\rho}_i)$, if $V(\widetilde{p}_i) \cap V(\widetilde{\rho}_i) \subseteq \{a\} \cup \{(\widetilde{\rho}_i)_0\}$ then we do nothing (take $P_i = \widetilde{P}_i$ and $S_i = \ell_i$). Otherwise there is some $0 < k < L(\widetilde{\rho}_i) - 1$ such that $(\widetilde{\rho}_i)_k \in V(\widetilde{p}_i)$. Let $i_0 < L(\widetilde{\rho}_i) - 1$ be the largest such index and let $j_0$ be the index of $\widetilde{p}_i$ such that $(\widetilde{\rho}_i)_{i_0} = (\widetilde{p}_i)_{j_0}$. In this case, take the "primary" loop $P_i$ to be the path $\widetilde{p}_i$ for its first $j_0$ vertices and then follow $\widetilde{\rho}_i$ back to $a$. Note that $P_i$ is simple by maximality of $i_0$. Next let $i_1$ be the least index of $\widetilde{\rho}_i$ so that $(\widetilde{\rho})_{i_1} \in \{(\widetilde{p}_i)_1, \dots, (\widetilde{p}_i)_{j_0}\}$ (it could be the case that $i_0 = i_1$). Finally let $i_2 < i_1$ be the index of $\widetilde{\rho}_i$ so that $(\widetilde{\rho}_i)_{i_2} \in V(\widetilde{p}_i)$ and the path from $(\widetilde{\rho}_i)_{i_2}$ to $(\widetilde{\rho}_i)_{i_1}$ is disjoint from $V(\widetilde{p}_i)$ (i.e. $i_2$ is the index of the meet of $\widetilde{\rho}_i$ and $\widetilde{p}_i$ just before $i_1$). Set $S_i$ to be the loop that follows $\widetilde{p}_i$ from $(\widetilde{\rho}_i)_{i_1}$ to $(\widetilde{\rho}_i)_{i_2}$ and then follows $\widetilde{\rho}_i$ back to $(\widetilde{\rho}_i)_{i_1}$. This loop is simple (by construction), does not contain $a$ and the graph $P_i \cup S_i$ has exactly one vertex with outgoing degree two. In this case we modify the triple $(\widetilde{p}_i, \widetilde{\rho}_i, \ell_i)$ to be the graph $P_i \cup S_i$.




The original triple $(\ell_i, \widetilde{p}_i, \widetilde{\rho}_i)$




The modified graph

Finally let $\mathcal{H}$ be the graph obtained by taking all edges and vertices in the loops $P_i$ and $S_i$ (and the edge $a \to a$ using lemma 5.1), which is a double bouquet. By lemma 5.3 there is a transient potential on $X_{\mathcal{H}}$ and hence also on $X_{\mathcal{G}}$. $\square$

5.3. **Simple Loops with Unbounded Size.** The main result of this section is lemma 5.6 which shows that if $\mathcal{G}$ has simple loops of arbitrarily large size based at a common vertex, then $X_{\mathcal{G}}$ has a transient potential.

For the construction that we will perform, we will need some notation.

**Definition 5.3.** *For some fixed $a \in \mathcal{S}_{\mathcal{G}}$ and a collection of simple loops $\ell, \ell_1, \ell_2, \ldots, \ell_n$ based at $a$, we say that $\ell$ is* semi-disjoint *from the collection $\{\ell_1, \ldots, \ell_n\}$ if there exists $i = i(\ell)$ and non-negative integers $A(\ell, \ell_i), B(\ell, \ell_i)$ such that*

- $A(\ell, \ell_i) < L(\ell) - B(\ell, \ell_i) - 1$;
- $(\ell)_j = (\ell_i)_j$ *for all* $0 \leq j \leq A(\ell, \ell_i)$;
- $(\ell)_{L(\ell)-j-1} = (\ell_i)_{L(\ell_i)-j-1}$ *for all* $0 \leq j \leq B(\ell, \ell_i)$;
- $(\ell)_j \notin \big(V(\ell_1) \cup \cdots \cup V(\ell_n)\big)$ *for any* $A(\ell, \ell_i) < j < L(\ell) - B(\ell, \ell_i) - 1$.

For a simple loop $\ell$, if $l$ is another simple loop then we define

$$\begin{aligned} A^*_\ell(l) &= \max\{i \mid (\ell)_j = (l)_j \text{ for all } j \leq i\} \\ B^*_\ell(l) &= \max\{i \mid (\ell)_{L(\ell)-j-1} = (l)_{L(l)-j-1} \text{ for all } j \leq i\}. \end{aligned}$$

The statement that $\ell$ is *not* semi-disjoint from $l$ means exactly that there is some $i$ with $A^*_\ell(l) < i < L(\ell) - B^*_\ell(l) - 1$ such that $(\ell)_i = (l)_i$.

For fixed $a \in \mathcal{S}_{\mathcal{G}}$, a simple loop $\ell$ based at $a$ determines a linear ordering on its vertices:

$$a = (\ell)_0 < (\ell)_1 < \cdots < (\ell)_{L(\ell)-2}$$

where we take $a$ to be the *least* element in this ordering rather than the greatest. We denote the ordering determined by $\ell$ with $\leq_\ell$.

Note that for a sequence of simple loops $p_1, p_2, \ldots$ based at a common point $a \in \mathcal{S}_{\mathcal{G}}$, exactly one of the following three conditions holds:

C1: There are infinitely many $k$ such that $p_k$ is semi-disjoint from $\{p_1\}$.
C2: C1 does not hold, but for infinitely many $k$, the set $V(p_1) \cap V(p_k)$ is ordered in the same way by $\leq_{p_1}$ and $\leq_{p_k}$.
C3: C1 and C2 do not hold and there exist distinct vertices $v_1, v_2$ in $p_1$ such that $v_1 \leq_{p_1} v_2$ but for infinitely many $k$, $v_2 \leq_{p_k} v_1$.

The following is a technical lemma which will allow us to prove lemma 5.6.

**Lemma 5.5.** *Suppose $\mathcal{G}$ does not have an infinite ray. If there is a vertex $a \in \mathcal{S}_{\mathcal{G}}$ and a sequence $p_1, p_2, \ldots$ of simple loops based at $a$ with $L(p_{i+1}) > L(p_i)$ $\forall i$, then there exist integers $K_1, K_2$ and a subsequence $p_{a_0}, p_{a_1}, p_{a_2}, \ldots$ such that $p_{a_k}$ is semi-disjoint from $\{p_{a_0}\}$, $A(p_{a_k}, p_{a_0}) = K_1$ and $B(p_{a_k}, p_{a_0}) = K_2$ for all $k \geq 1$.*

*Proof.* There are three cases to consider.

*Case 1*: If the list $p_1, p_2, \ldots$ has property C1 then find integers $K_1, K_2$ such that for infinitely many $k$, $p_k$ is semi-disjoint from $p_1$ and $A(p_k, p_1) = K_1$, $B(p_k, p_1) = K_2$ (recall that $A(p_k, p_1)$, $B(p_k, p_1)$ can take on only finitely many different values since they correspond to vertices in $p_1$). Pass to the subsequence including only $p_1$ and loops of this type.

*Case 2*: If the list $p_1, p_2, \dots$ has property C2, we claim that a subsequence has property C1. Begin by finding integers $K_1', K_2'$ such that $A^*_{p_1}(p_k) = K_1'$ and $B^*_{p_1}(p_k) = K_2'$ for infinitely many $k$ (again $A^*_{p_1}(p_k)$, $B^*_{p_1}(p_k)$ can take on only finitely many different values). Define

$$I := \{1\} \cup \{k \mid A^*_{p_1}(p_k) = K_1' \text{ and } B^*_{p_1}(p_k) = K_2'\}$$

to be the index set of the subsequence of loops of this type (and also $p_1$).

For each $k \in I$ define

$$\begin{aligned} \{i_1^k, \dots, i_{r(k)}^k\} &:= \{i \mid (p_k)_i \in V(p_1)\} \\ \{j_1^k, \dots, j_{r(k)}^k\} &:= \{j \mid (p_1)_j \in V(p_k)\}, \end{aligned}$$

ordered so that $(p_k)_{i_n^k} = (p_1)_{j_n^k}$ for all $n$. Then for each $k$, there is some $m$ such that the (simple) loop:

$$a \xrightarrow{[p_1]} (p_1)_{j_m^k} = (p_k)_{i_m^k} \xrightarrow{[p_k]} (p_k)_{i_{m+1}^k} = (p_1)_{j_{m+1}^k} \xrightarrow{[p_1]} a$$

has length greater than or equal to $\frac{L(p_k)}{L(p_1)}$ (this is the loop $p_1$ with one "excursion" through part of $p_k$); where $v_1 \xrightarrow{[p]} v_2$ denotes the segment of path $p$ from $v_1$ to $v_2$. This is possible to do because the segment $(p_k)_{i_m^k} \xrightarrow{[p_k]} (p_k)_{i_{m+1}^k}$ must have length at least $\frac{L(p_k)}{L(p_1)}$ for some $m$ (since the entire path $p_k$ is made up of segments of this form). Denote this loop by $\widetilde{p}_k$. Note that $\widetilde{p}_k$ is semi-disjoint from $p_1$ and $A(\widetilde{p}_k, p_1) \geq A^*_{p_k}(p_1)$ (similarly for the $B$'s). Pass to a subsequence of $p_1, \widetilde{p}_2, \widetilde{p}_3, \dots$ (that includes $p_1$) for which the lengths of the paths are strictly increasing. This subsequence has property C1.

*Case 3*: If the list $p_1, p_2, \dots$ has property C3 we first ask whether one of the subsequences $p_k, p_{k+1}, p_{k+2}, \dots$ (just the indices greater than or equal to $k$) has either property C1 or C2. If so, then we are reduced to one of the previous cases on this subsequence. Otherwise, for every $k$, there are distinct vertices $b_k, c_k \in V(p_k)$ such that $b_k \leq_{p_k} c_k$ but $c_k \leq_{p_i} b_k$ for infinitely many $i$. We inductively define a nested collection of subsequences of $\{p_k\}_{k=1}^{\infty}$. Let

$$I_1 := \{j \in \mathbb{N} \mid c_1 \leq_{p_j} b_1\}$$

and let $s_1$ be the least element of $I_1$. Now suppose we have constructed *infinite* sets

$$I_1 \supseteq \cdots \supseteq I_r$$

such that $I_i := \{j \in I_{i-1} \mid j > s_{i-1} \text{ and } c_{s_{i-1}} \leq_{p_j} b_{s_{i-1}}\}$ where $s_{i-1}$ is the least element of $I_{i-1}$ (here we take $I_0 := \mathbb{N}$). If there are infinitely many $k \in I_r$ such that $c_{s_r} \leq_k b_{s_r}$ then let $I_{r+1} := \{j \in I_r \mid c_{s_r} \leq_{p_j} b_{s_r}\}$. Otherwise the construction terminates.

If the construction terminates after $N$ steps then the subsequence $\{p_j \mid j \in I_N\}$ has either property C1 or C2, whence a sub-subsequence is of type C1 (and the lemma follows). Otherwise the process continues indefinitely and we obtain a sequence of index sets

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

and integers $s_1, s_2, s_3, \dots$ (recall $s_i$ is the least element in $I_i$) so that

$$c_{s_k} \leq_{p_j} b_{s_k} \text{ for all } j \in I_{k+1}.$$

Note that $s_{k+1} > s_k$ and $b_{s_k} \leq_{s_k} c_{s_k}$ by definition. The remainder of the proof is to show that, under these conditions, there is an infinite forward or backward ray contained in $\mathcal{G}$, in contradiction to the assumptions of the lemma.

**Claim.** *At least one of the sets* $\{b_{s_1}, b_{s_2}, \dots\}$, $\{c_{s_1}, c_{s_2}, \dots\}$ *is infinite.*

*Proof.* We begin by noting that if $\{b_{s_i}, c_{s_i}\} = \{b_{s_j}, c_{s_j}\}$ and $i < j$ then either $(b_{s_i}, c_{s_i}) = (b_{s_j}, c_{s_j})$ or $(b_{s_i}, c_{s_i}) = (c_{s_j}, b_{s_j})$ (as ordered lists). The first case is impossible since $b_{s_j} \leq_{p_{s_j}} c_{s_j}$ but $c_{s_i} \leq_{p_{s_j}} b_{s_i}$ which implies that $b_{s_i} = c_{s_i}$; a contradiction. The second case is also impossible since $c_{s_i} \leq_{p_{s_{j+1}}} b_{s_i}$ and $c_{s_j} \leq_{p_{s_{j+1}}} b_{s_j}$ gives a similar contradiction.

On the other hand, if both $\{b_{s_1}, b_{s_2}, \dots\}$ and $\{c_{s_1}, c_{s_2}, \dots\}$ are finite (say all distinct elements are exhausted in both sets by index $r$) then the set $\{b_{s_{r^2+1}}, c_{s_{r^2+1}}\}$ must be equal to $\{b_i, c_i\}$ for some $1 \leq i \leq r^2$. This establishes the claim.

Without loss of generality, suppose the set $\{b_{s_1}, b_{s_2}, \dots\}$ is infinite and, by passing to a subsequence, assume that $b_{s_i} \neq b_{s_j}$ for $i \neq j$. Recall from the construction of the sets $I_j$ that

$$b_{s_1}, \dots, b_{s_n} \in V(p_{s_n}) \text{ for every } n.$$

Therefore the set $\{b_{s_1}, b_{s_2}, \dots\}$ has the property that if one of its elements $b_{s_i}$ occurs in some path $p_{s_j}$, it also occurs in $p_{s_k}$ for every $k \geq j$. This set continues to have this property if we pass to any subsequence of the paths $\{p_{s_i}\}_{i=1}^{\infty}$. Our goal is to produce a subsequence of $\{p_{s_i}\}_{i=1}^{\infty}$, say $p_{a_1}, p_{a_2}, \dots$ with the property that for any $k$ and any $v \in V(p_{a_k})$ either $v \in V(p_{a_n})$ for all $n > k$ or $v \notin V(p_{a_n})$ for any $n > k$. We will then produce a set

$$\mathcal{A} = \bigcup_{k=1}^{\infty} \{v \in V(p_{a_k}) \mid v \in V(p_{a_n}) \text{ for all } n \geq k\}$$

which contains all vertices of the first type (this set will be infinite since $b_{s_i} \in \mathcal{A}$ for all $i$) and use it to construct an infinite ray in $\mathcal{G}$.

We begin by producing the subsequence – this is done one path at a time. Enumerate the vertices of $p_{s_1}$ by $v_i = (p_{s_1})_i$. Define a set of indices for $v_1$ as follows. If $v_1 \in V(p_{s_k})$ for only finitely many $k > 1$ then let $J_{v_1}$ be the set of $s_k$ for which $v_1 \notin V(p_{s_k})$. Otherwise $v_1 \in V(p_{s_k})$ for infinitely many $k > 1$; in this case let $J_{v_1}$ be the set of $s_k$ for which $v \in V(p_{s_k})$. Now suppose we have constructed infinite sets (i.e. subsequences)

$$J_{v_1} \supseteq \cdots \supseteq J_{v_r}$$

so that for every $1 \leq i \leq r$, either $v_i \in V(p_j)$ for all $j \in J_{v_i}$ or $v_i \notin V(p_j)$ for any $j \in J_{v_i}$. If $r < L(p_{s_1}) - 1$ then

$$J_{v_{r+1}} := \begin{cases} \{j \in J_{v_r} \mid v_{r+1} \notin V(p_j)\} \text{ if } v_{r+1} \in V(p_{s_k}) \text{ for only finitely many } k \in J_{v_r} \\ \{j \in J_{v_r} \mid v_{r+1} \in V(p_j)\} \text{ otherwise.} \end{cases}$$

Define $J := \{s_1\} \cup J_{v_{L(p_{s_1})-1}}$ (notice that $s_1$ is the least element of $J$). Then for every $v \in V(p_{s_1})$, either $v \in V(p_i)$ for all $i \in J \setminus \{s_1\}$ or $v \notin V(p_i)$ for any $i \in J \setminus \{s_1\}$ (that is, the vertices in $p_{s_1}$ occur either in all subsequent paths in this subsequence or in none of them). Set $a_1 := s_1$.

Next, let $a_2 := \min\{i \in J \mid i > a_1\}$, enumerate the vertices in $p_{a_2}$, and perform the above procedure for $p_{a_2}$: namely produce an infinite subset $J' \subseteq J$ so that $a_1, a_2 \in J'$ and for each $v \in V(p_{a_2})$, either $v \in V(p_i)$ for all $i \in J' \setminus \{a_1, a_2\}$ or

$v \notin V(p_i)$ for any $i \in J' \setminus \{a_1, a_2\}$. Continue this process inductively to obtain a subsequence with index set $\mathcal{J} = \{a_1, a_2, \dots\}$ so that for any $k$, if $v \in V(p_{a_k})$ then either $v \in V(p_{a_m})$ for all $m \geq k$ or $v \notin V(p_{a_m})$ for all $m \geq k$. We draw special attention to the first type of vertices by setting

$$\{d_1^k, \dots, d_{n_k}^k\} := \{v \in V(p_{a_k}) \mid v \in V(p_{a_m}) \text{ for all } m \geq k\},$$

where $n_k$ is the number of elements in the set on the right. Then the set

$$\mathcal{A} := \{d_j^i \mid i \in \mathbb{N}, j \leq n_i\} \supset \{b_{s_1}, b_{s_2}, \dots\}$$

is infinite. We will use $\mathcal{A}$ to produce an infinite ray. In order to do this, we introduce an ordering on the elements of $\mathcal{A}$; this is done inductively as follows.

Since $d_1^1, \dots, d_{n_1}^1 \in V(p_{a_k})$ for all $k \geq 1$ and this (finite) set has only finitely many possible orderings, there is a permutation $\sigma$ such that

$$d_{\sigma(1)}^1 \leq_{p_{a_k}} d_{\sigma(2)}^1 \leq_{p_{a_k}} \cdots \leq_{p_{a_k}} d_{\sigma(n_1)}^1$$

for infinitely many $k \in \mathbb{N}$. Let $\mathcal{K}_1$ be the set of indices for which this is true, then all of the orderings $\{\leq_{p_j} \mid j \in \mathcal{K}_1\}$ agree on the set $\{d_1^1, \dots, d_{n_1}^1\}$.

Now inductively construct infinite sets $\mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq \cdots$ so that for every $i$ the orderings $\{\leq_j \mid j \in \mathcal{K}_i\}$ agree on the set

$$\{d_1^1, \dots, d_{n_1}^1, \dots, d_1^i, \dots, d_{n_i}^i\}$$

($\mathcal{K}_{i-1}$ has infinitely many elements and $\{d_1^1, \dots, d_{n_1}^1, \dots, d_1^i, \dots, d_{n_i}^i\}$ has only finitely many orderings). Finally let $\leq_\infty$ be an ordering of $\mathcal{A}$ which, for any $j \in \mathcal{K}_i$ is equivalent to $\leq_{p_j}$ on $\{d_1^1, \dots, d_{n_1}^1, \dots, d_1^i, \dots, d_{n_i}^i\}$.

Suppose there is a sequence of distinct vertices $d_1, d_2, \dots \in \mathcal{A}$ such that

$$d_1 \leq_\infty d_2 \leq_\infty \cdots .$$

For each $i$, find $j(i)$ so that $\{d_1, \dots, d_i\} \subseteq \{d_1^1, \dots d_{n_1}^1, \dots, d_1^{j(i)}, \dots, d_{n_{j(i)}}^{j(i)}\}$. Let $e_i \in \mathcal{K}_{j(i)}$ (chosen so that $e_1 < e_2 < \cdots$). Then $d_1, \dots, d_i \in V(p_{e_i})$ and $\leq_{p_{e_i}}$ agrees with $\leq_\infty$ on $\{d_1, \dots, d_i\}$. Then form the path

$$a \xrightarrow{[p_{e_1}]} d_1 \xrightarrow{[p_{e_2}]} d_2 \xrightarrow{[p_{e_3}]} d_3 \to \cdots \tag{5.1}$$

where if $v_1, v_2 \in V(p_{e_i})$ then $v_1 \xrightarrow{[p_{e_i}]} v_2$ denotes segment of $p_{e_i}$ from $v_1$ to $v_2$.

We claim that any vertex in this path repeats at most finitely many times. Indeed suppose $v$ is a vertex that is encountered on the section $d_i \xrightarrow{[p_{e_{i+1}}]} d_{i+1}$. If $v \notin \mathcal{A}$, then $v \notin V(p_{e_j})$ for any $j > i+1$ (so $v$ repeats at most finitely many times). Otherwise $v \in \mathcal{A}$ and there are two possibilities: either $d_j \leq_\infty v$ for all $j$ or there is some $j$ so that $v \leq_\infty d_j$. In the first case ($d_j \leq_\infty v$ for all $j$), since $v \in \mathcal{A}$ there is some $N$ so that $v \in V(p_{e_n})$ for all $n > N$ and some $M \geq N$ so that $d_{m+1} \leq_{p_{e_{m+1}}} v$ for all $m > M$ (we take $M$ so that $d_{M+1} \leq_\infty v$ implies that $d_{M+1} \leq_{p_{e_{M+1}}} v$). In this case $v$ cannot occur in the segment $d_m \xrightarrow{[p_{e_{m+1}}]} d_{m+1}$ for any $m > M$, so in the entire path (5.1) it occurs at most finitely many times. In the second case (there is some $j$ so that $v \leq_\infty d_j$), we have $v \leq_{p_{e_i}} d_j \leq_{p_{e_i}} d_i$ for all sufficiently large $i$ (i.e. large enough that $\leq_\infty$ agrees with $\leq_{e_i}$ on the set $\{v, d_j, d_i\}$). Again $v$ cannot occur in the segment $d_i \xrightarrow{[p_{e_i}]} d_{i+1}$ for such $i$, so it occurs at most finitely many times in (5.1).

Since every vertex repeats at most finitely many times in this path, it has a subpath that is an infinite forward ray which contradicts the assumptions of the lemma.

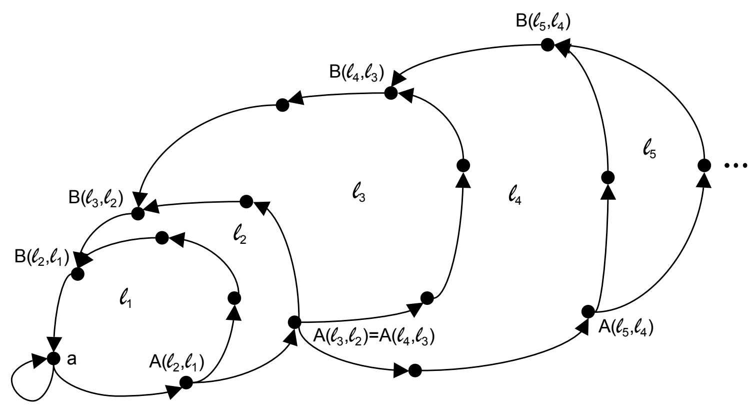
Otherwise there is a sequence $d_1, d_2, \dots \in A_\infty$ such that

$$d_1 \geq_\infty d_2 \geq_\infty \cdots$$

We argue as above to produce a backwards infinite ray and obtain a similar contradiction. □

**Lemma 5.6.** *If there is a vertex $a \in \mathcal{S}_\mathcal{G}$ such that there are simple loops based at $a$ of arbitrarily large size, then $X_\mathcal{G}$ has a transient Markov potential.*

*Proof.* By lemma 5.2 it is enough to consider the case when $\mathcal{G}$ does not have an infinite ray. Our goal is to produce an infinite sequence of loops $\ell_0, \ell_1, \dots$ such that for every $n$, $\ell_n$ is semi-disjoint from $\{\ell_1, \dots, \ell_{n-1}\}$, $i(\ell_n) = n-1$, and the sequences $\{A(\ell_n, \ell_{n-1})\}_n$ and $\{B(\ell_n, \ell_{n-1})\}_n$ are (not necessarily strictly) increasing (see definition of semi-disjioint to recall these numbers). This situation is pictured below.




The subgraph $\mathcal{H}$ (to be constructed)

Enumerate the simple loops based at $a$ by $p_1, p_2, \dots$ By passing to a subsequence, we may assume that $L(p_{n+1}) > L(p_n)$ for every $n$. We proceed to construct the sequence $\{\ell_k\}_{k=0}^\infty$ by induction.

Apply lemma 5.5 to the sequence $p_1, p_2, \dots$ to obtain integers $K_1^1, K_2^1$ and an index set $\mathcal{A}_1 \subseteq \mathbb{N}$ (with minimal element $a_1$) so that $p_k$ is semi-disjoint from $\{p_{a_1}\}$, $A(p_k, p_{a_1}) = K_1^1$ and $B(p_k, p_{a_1}) = K_2^1$, for every $k \in \mathcal{A}_1 \setminus \{a_1\}$. Set $\ell_1 := p_{a_1}$.

Next apply lemma 5.5 to the sequence $\{p_i \mid i \in \mathcal{A}_1 \setminus \{a_1\}\}$ to obtain integers $K_1^2, K_2^2$ and an index set $\mathcal{A}_2 \subseteq \mathcal{A}_1$ (with minimal element $a_2$) so that $p_k$ is semi-disjoint from $\{p_{a_2}\}$, $A(p_k, p_{a_2}) = K_1^2$ and $B(p_k, p_{a_2}) = K_2^2$ for every $k \in \mathcal{A}_2 \setminus \{a_2\}$. Set $\ell_2 := p_{a_2}$. Note that $K_1^2 \geq K_1^1$ since for any $k \in \mathcal{A}_2 \subseteq \mathcal{A}_1$, $p_k$ follows $\ell_1$ for the first $K_1^1$ vertices whence also $\ell_2$ (since *it* follows $\ell_1$ for the first $K_1^1$ vertices). Similarly $K_2^2 \geq K_2^1$. Thus, for any $k \in \mathcal{A}_2 \setminus \{a_2\}$, $p_k$ is semidisjoint from $\{\ell_1, \ell_2\}$ and $i(p_k) = 2$. Continue this process inductively to construct loops $\ell_1, \dots, \ell_r$ and infinite sets of indices

$$\mathcal{A}_1 \supseteq \cdots \supseteq \mathcal{A}_r$$

so that

- $\ell_i$ is semi-disjoint from $\{\ell_1, \dots, \ell_{i-1}\}$ for every $2 \leq i \leq r$

- $i(\ell_i) = i-1$ for every $2 \le i \le r$ and the sequences $\{A(\ell_n, \ell_{n-1})\}_n$ and $\{B(\ell_n, \ell_{n-1})\}_n$ are nondecreasing.

Thus we obtain our desired sequence by induction. So, the graph $\mathcal{H}$ obtained by taking all of the vertices and edges in $\{\ell_i\}_{i=0}^{\infty}$ (and $a \to a$ by lemma 5.1) is of the desired type.

We now show how to define a potential with finite discriminant on $X_{\mathcal{H}}$. Define

$$\phi(x_0, x_1) := \begin{cases} \log\left(\frac{1/n^2}{\sum_{k=1}^{n-1} 1/k^2}\right) & \text{if } x_0 = (\ell_n)_{K_1^n} \text{ and } x_1 = (\ell_n)_{K_1^n+1} \\ 0 & \text{otherwise.} \end{cases}$$

By construction, the only first return loops on $\mathcal{H}$ are the loops $\ell_i$ (and $a \to a$) and the contribution of each loop $\ell_n$ to the sum defining $r_\phi(x)$ is $\asymp \frac{1}{n^2} x^{L(\ell_n)}$ (it is no smaller than $\frac{1}{\zeta(2)n^2} x^{L(\ell_n)}$ and no larger than $\frac{1}{n^2} x^{L(\ell_n)}$). Therefore $r_\phi(x)$ converges at $x = 1$ which is its radius of convergence. Therefore $\phi$ has finite discriminant and so there exists a transient potential on $X_{\mathcal{H}}$ and so also on $X_{\mathcal{G}}$. □

## 6. Proof of Theorem 2.1

By theorem 3.1 a graph with a finite uniform Rome cannot have a transient potential. We now prove the converse.

Suppose $X_{\mathcal{G}}$ has no transient potentials then:

- $\mathcal{G}$ cannot have infinitely many disjoint simple loops (Lemma 5.4);
- For any $a \in \mathcal{S}_{\mathcal{G}}$ the length of the simple loops based at $a$ is bounded. (Lemma 5.6)

We will show that any graph that has these two properties has a finite uniform Rome.

Enumerate all simple loops in $\mathcal{G}$ as $\ell_1, \ell_2, \ell_3, \dots$ (ignoring the base point). Since there are not infinitely many disjoint simple loops in $\mathcal{G}$, there must be some collection $\ell_{k_1}, \dots, \ell_{k_r}$ such that for any $i$ there exists $j$ with $V(\ell_i) \cap V(\ell_{k_j}) \neq \emptyset$. Define

$$F := V(\ell_{k_1}) \cup \dots \cup V(\ell_{k_r}).$$

Observe that $F$ is a finite set and every simple loop in $\mathcal{G}$ has at least one vertex in $F$. For each $v \in F$ the lengths of the simple loops based at $v$ are bounded and $F$ is finite, so there is some $N \in \mathbb{N}$ such that every simple loop in $\mathcal{G}$ has length less than $N$.

We claim that $F$ is a uniform Rome for $\mathcal{G}$. Consider an arbitrary path $p$ in $X \setminus F$

$$p : b_0 \to b_1 \to b_2 \to b_3 \to \dots \to b_n$$

We aim to bound the size of $n$ (so that all paths in $X \setminus F$ have uniformly bounded length). Our first observation is that $p$ is simple since if any vertex repeats then $p$ contains a simple loop and, therefore, at least one vertex in $F$. Let $i_1$ be the least index so that there is a path

$$p_1 : b_n \xrightarrow{L(p_1)} b_{i_1}$$

with $(p_1)_j \notin \{b_0, \dots, b_n\}$ for any $0 < j < L(p_1) - 1$ (note that $i_1 < n$ by topological mixing). Let $i_2$ be the least index so that there is a path from one of the vertices $b_{i_1}, \dots, b_{n-1}$ to $b_{i_2}$:

$$p_2 : \{b_{i_1}, \dots, b_{n-1}\} \xrightarrow{L(p_2)} b_{i_2}$$

with $(p_2)_j \notin \{b_0, \ldots, b_n\}$ for any $0 < j < L(p_2) - 1$. Note that $i_2 < i_1$ since there is a path from $b_n$ to $b_0$ and it must pass through the set $\{b_{i_1}, \ldots, b_{n-1}\}$ before it can get to any of the vertices $\{b_0, \ldots, b_{i_1-1}\}$ (by choice of $i_1$). Now suppose we have constructed indices $i_1 > \cdots > i_k$ and paths $p_1, \ldots, p_k$ so that $i_k$ is the least index such that there is a path

$$q : \{b_{i_{k-1}}, \ldots, b_{i_{k-2}-1}\} \xrightarrow{L(q)} b_{i_k}$$

with $(q)_j \notin \{b_0, \ldots, b_n\}$ for any $0 < j < L(q) - 1$, and where $p_k$ is such a path. If $i_k > 0$ then let $i_{k+1}$ be the least index such that there is a path

$$q : \{b_{i_k}, \ldots, b_{i_{k-1}-1}\} \xrightarrow{L(q)} b_{i_{k+1}}$$

with $(q)_j \notin \{b_0, \ldots, b_n\}$ for any $0 < j < L(q) - 1$. Note that $i_{k+1} < i_k$ since there is a path from $b_n$ to $b_0$ and it must pass through the set $\{b_{i_k}, \ldots, b_{i_{k-1}-1}\}$ before it can get to any of the vertices $\{b_0, \ldots, b_{i_k-1}\}$. Continue this process as long as $i_k > 0$.

This process cannot continue indefinitely, so let $i_1 > \cdots > i_r = 0$ and $p_1, \ldots, p_r$ be the indices and paths so constructed. Observe that if $|i - j| > 1$ then $V(p_i) \cap V(p_j) = \emptyset$ since if $i > j + 1$ then $V(p_i) \cap V(p_j)$ cannot contain any vertices in $\{b_0, \ldots, b_n\}$ (by definition of the indices $i_1, \ldots, i_r$) and cannot contain any vertices not in the path $p$ (otherwise we could follow the path $p_i$ until it meets $p_j$ and then follow $p_j$ to obtain a path that violates minimality of the indices $i_1, \ldots, i_r$). Therefore the simple loops

$$\ell_k : b_{i_k} \to b_{i_k+1} \to \cdots \to b_{i_{k-1}} \xrightarrow{[p_k]} b_{i_k}$$

have $V(\ell_{2n}) \cap V(\ell_{2m}) = \emptyset$ for all $n \neq m$ (similarly for the odd-indexed loops). Moreover, every vertex in $\{b_0, \ldots, b_n\}$ is contained in one of these loops. But the total number of vertices in these (two sets of pairwise disjoint, simple) loops is bounded, so we get

$$(n + 1) \leq 2N \cdot (\text{maximal number of pairwise disjoint loops}) \leq 2N \cdot |F| .$$

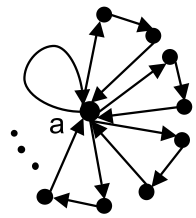
Therefore $X$ has a finite uniform Rome. □

## 7. Appendix A

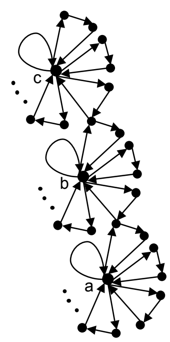
Here we give examples of graphs with finite uniform Romes (see Definition 1.1). First note that *any* graph has a uniform Rome by taking $F = \mathcal{S}$ and $N = 1$. On the other hand, it is a strong restriction on $\mathcal{G}$ to require a *finite* uniform Rome.

**Example 1**: Any finite graph has a finite uniform Rome by taking $F = \mathcal{S}$ and $N = 1$ (so subshifts of finite type cannot have transient potentials).

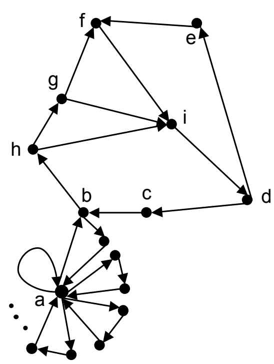
**Example 2**: A topologically mixing infinite bouquet of first return loops of length less than or equal to a fixed integer. Below is pictured an infinite bouquet of first return loops of length three (and an edge $a \to a$). We can take $F = \{a\}$ and $N = 2$.




**Example 3**: Finitely many bouquets can be joined together (see below). Here we take $F = \{a, b, c\}$ and $N = 3$.




**Example 4**: A finite graph can be joined to an infinite bouquet. Here take $F = \{a, b, c, d, e, f, g, h, i\}$ and $N = 2$.




## 8. Appendix B

The following result follows from the discussion of the symmetric random walk on $\mathbb{Z}$ in [5] (p. 73-78). Let $\mathcal{G}(\mathbb{N}_0)$ be the graph obtained from $\mathbb{N} \cup \{0\}$ by putting a directed edge from $0 \to 0$ as well as directed edges of the form $n \to (n+1)$ and $(n+1) \to n$ for all $n \geq 0$.

**Lemma 8.1.** *The potential* $\phi(x_0, x_1) = -\log(2)$ *on* $X_{\mathcal{G}(\mathbb{N}_0)}$ *has finite discriminant.*

*Proof.* For the symmetric random walk on $\mathbb{Z}$, the probability that a trajectory starting at zero returns at time $2n$ is

$$p_{2n} = \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \asymp \frac{1}{\sqrt{\pi n}}$$

by Stirling's formula. Let $f_{2n}$ be the probability that a trajectory starting at zero will return for the first time at time $2n$. Then $f_{2n} = p_{2n-2} - p_{2n} = \frac{1}{2n-1} \cdot p_{2n}$ [5, Lemma 1] (p. 76). Therefore, the probability that a trajectory starting at zero will stay in $\mathbb{Z}^+$ and return for the first time at time $2n$ is $\asymp \frac{1}{(4n-2)\sqrt{\pi n}}$. On the other hand, this probability is exactly $(\frac{1}{2})^{2n} \cdot \#$(paths in $\mathbb{Z}^+$ that first return at time $2n$).

Now consider the potential $\phi$ on $\mathcal{G}(\mathbb{N}_0)$. We have

$$Z^*_{2n}(\phi, 0) = \left(\frac{1}{2}\right)^{2n} \cdot \#(\text{first return loops of length } 2n \text{ based at } 0 \text{ in } \mathcal{G}(\mathbb{N}_0)) \asymp \frac{C}{n^{3/2}}$$

for some constant $C$. Moreover, $Z^*_1(\phi, 0) = \frac{1}{2}$ and $Z^*_{2n+1}(\phi, 0) = 0$ for all $n > 0$ so $r_\phi(1) < \infty$ and $r_\phi(x) = \infty$ for all $x > 1$; that is, $\phi$ has finite discriminant and $P_G(\phi) = 0$. □

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