FREE ERGODIC $\mathbb{Z}^2$-SYSTEMS AND COMPLEXITY

VAN CYR AND BRYNA KRA

Abstract. Using results relating the complexity of a two dimensional subshift to its periodicity, we obtain an application to the well-known conjecture of Furstenberg on a Borel probability measure on $[0, 1)$ which is invariant under both $x \mapsto px \pmod 1$ and $x \mapsto qx \pmod 1$, showing that any potential counterexample has a nontrivial lower bound on its complexity.

1. Introduction

1.1. **Complexity and periodicity.** For a one dimensional symbolic system (X, σ), meaning that $X \subset \mathcal{A}^{\mathbb{Z}}$ is a closed set, where $\mathcal{A}$ is a finite alphabet, that is closed under the left shift $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, the Morse-Hedlund Theorem gives a simple relation between the complexity of the system and periodicity. Namely, if $P_X(n)$ denotes the complexity function, which counts the number of nonempty cylinder sets of length n in X, then (X, σ) is periodic if and only if there exists $n \in \mathbb{N}$ such that $P_X(n) \leq n$. Both periodicity and complexity have natural generalizations to higher dimensional systems. For example, for a two dimensional system (X, σ, τ), meaning that $X \subset \mathcal{A}^{\mathbb{Z}^2}$ is a closed set that is invariant under the left and down shifts $\sigma, \tau\colon \mathcal{A}^{\mathbb{Z}^2} \to \mathcal{A}^{\mathbb{Z}^2}$, the two dimensional complexity $P_X(n, k)$ is the number of nonempty n by k cylinder sets. In a partial solution to Nivat's Conjecture [11], the authors [2] showed that if (X, σ, τ) is a transitive $\mathbb{Z}^2$-subshift and there exist $n, k \in \mathbb{N}$ such that $P_X(n, k) \leq nk/2$, then there exists $(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $\sigma^i\tau^j x = x$ for all $x \in X$. In this note, we give an application of this theorem to Furstenberg's well-known "$\times p, \times q$ problem."

1.2. **The $\times p, \times q$ problem.** Let $S, T\colon [0, 1) \to [0, 1)$ denote the maps $Sx := px \pmod 1$ and $Tx := qx \pmod 1$, where $p, q \geq 1$ are multiplicatively independent integers (meaning that p and q are not both powers of the same integer). In the 1960's, Furstenberg [4] proved that any closed subset of $[0, 1)$ that is invariant under both S and T is either all of $[0, 1)$ or is finite. He asked whether a similar statement holds for measures:

Conjecture 1.1 (Furstenberg). *Let μ be a Borel probability measure on $[0, 1)$ that is invariant under both S and T and is ergodic for the joint action of S and T. Then either μ is Lebesgue measure or μ is atomic.*

Progress was made in the 1980's with the work of Lyons [9], followed soon thereafter by Rudolph's proof that positive entropy $h_\mu(\cdot)$ of the measure μ with respect to one of the transformations implies the result for relatively prime p and q. This was generalized to multiplicatively independent integers by Johnson:

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Theorem 1.2 (Rudolph [16] and Johnson [6]). *Let μ be a Borel probability measure on $[0,1)$ that is invariant under both S and T and is ergodic for the joint action of S and T. If $h_\mu(S) > 0$ (or equivalently $h_\mu(T) > 0$), then μ is Lebesgue measure.*

One way to interpret this theorem is that the set of $\langle S,T\rangle$-ergodic measures experiences an entropy gap with respect to the one-dimensional action generated by S (or equivalently by T). Informally, if μ has *high entropy* (in this case meaning that $h_\mu(S) > 0$), then its entropy with respect to S is actually $\log p$ and μ is Lebesgue measure. Our main theorem is that the set of $\langle S,T\rangle$-ergodic measures also experiences a *complexity gap*, in a sense we make precise. We show (Theorem 1.9) that if μ has *low complexity* (meaning that a certain function grows subquadratically), then it actually has *bounded complexity* (meaning that this function is bounded) and μ is atomic. Moreover, all atomic measures have bounded complexity.

1.3. **Rephrasing $\times p, \times q$ in symbolic terms.** We begin by recasting Furstenberg's Conjecture and the Rudolph-Johnson Theorem as statements about symbolic dynamical systems. We start by setting some terminology and notation.

A *(measure preserving) system* $(X,\mathcal{X},\mu,G)$ is a measure space X with an associated σ-algebra $\mathcal{X}$, probability measure μ, and an abelian group G of measurable, measure preserving transformations. If the context is clear, we omit the σ-algebra from the notation, writing (X,μ,G), and call it a *system.* The system (X,μ,G) is *free* if the set $\{x\in X\colon gx = x\}$ has measure 0 for every $g\in G$ and the system is *ergodic* if the only sets invariant under the action of G have either trivial or full measure. It follows that if (X,μ,G) is an ergodic system with an abelian group G of transformations, then the action of G is free if $g_1^{n_1}\circ\ldots\circ g_k^{n_k}\neq \mathrm{Id}$ for any $g_1,\ldots,g_k\in G$ and $(n_1,\ldots,n_k)\neq(0,\ldots,0)$.

Two systems $(X_1,\mathcal{X}_1,\mu_1,G)$ and $(X_2,\mathcal{X}_2,\mu_2,G)$ are *(measure theoretically) isomorphic* if there exist $X_1'\in\mathcal{X}_1$ and $X_2'\in\mathcal{X}_2$ with $\mu_1(X_1') = \mu_2(X_2') = 1$ such that $gX_1'\subset X_1'$ for all $g\in G$ and $gX_2'\subset X_2'$ for all $g\in G$, and there is an invertible bimeasurable transformation $\pi\colon X_1'\to X_2'$ such that $\pi_*\mu_1 = \mu_2$ and $\pi g(x) = g\pi(x)$ for all $x\in X_1'$, $g\in G$.

We are particularly interested in the $\mathbb{Z}^2$-system generated by the two commuting, measure preserving transformations S and T. In this case, we write $(X,\mathcal{X},\mu,S,T)$ for the $\mathbb{Z}^2$-system.

A *(topological) system* (X,G), is a compact metric space X and a group G of homeomorphisms mapping X to itself. If it is clear from the context that we are referring to a topological system, we call (X,G) a *system.* A system is said to be *minimal* if for any $x\in X$, the orbit $\{gx\colon g\in G\}$ is dense in X. By the Krylov-Bogolyubov Theorem, every system (X,G) admits an invariant Borel probability measure and if this measure is unique, we say that (X,G) is *uniquely ergodic.* A system (X,G) is *strictly ergodic* if it is both minimal and uniquely ergodic.

Let $\mathcal{A}$ denote a finite alphabet and let $\mathcal{A}^{\mathbb{Z}^2}$ be the set of $\mathcal{A}$-colorings of $\mathbb{Z}^2$. For $x\in\mathcal{A}^{\mathbb{Z}^2}$ and $\vec{u}\in\mathbb{Z}^2$, we denote the element of $\mathcal{A}$ that x assigns to $\vec{u}$ by $x(\vec{u})$. With respect to the metric

$$d(x,y) := 2^{-\inf\{|\vec{u}|\colon x(\vec{u})\neq y(\vec{u})\}},$$

$\mathcal{A}^{\mathbb{Z}^2}$ is compact and the leftward and downward shift maps $\sigma, \tau \colon \mathcal{A}^{\mathbb{Z}^2} \to \mathcal{A}^{\mathbb{Z}^2}$ given by

$$(\sigma x)(i,j) := x(i+1,j), \tag{1}$$
$$(\tau x)(i,j) := x(i,j+1) \tag{2}$$

are homeomorphisms. A closed set $X \subset \mathcal{A}^{\mathbb{Z}^2}$ which is invariant under the joint action of $\langle\sigma,\tau\rangle$ is called a $\mathbb{Z}^2$-*subshift*. (The analogous definitions hold for $\mathbb{Z}^d$-subshifts.)

A uniquely ergodic topological system $(\widehat{X}, \nu, G)$ is said to be a *topological model* for the measure preserving system $(X, \mathcal{X}, \mu, G)$ if there exists a measure theoretic isomorphism between $(\widehat{X}, \nu, G)$ and (X, μ, G). Again, we are mainly interested in topological systems generated by two transformations, and in this case we denote the topological system by $(\widehat{X}, \sigma, \tau)$.

The Jewett-Krieger Theorem [5, 8] states that any ergodic $\mathbb{Z}$-system has a strictly ergodic topological model, meaning that the system is measure theoretically isomorphic to a minimal, uniquely ergodic topological system. This was generalized to cover ergodic $\mathbb{Z}^d$-systems by Weiss [18], and further refined by Rosenthal (we only state it for $\mathbb{Z}^2$, as this is the only case relevant for our purposes):

Theorem 1.3 (Rosenthal [15])**.** *Let $(X, \mathcal{X}, \mu, S, T)$ be an ergodic, free $\mathbb{Z}^2$-system with entropy less than $\log k$. Then there exists a minimal, uniquely ergodic subshift $\widehat{X} \subset \{1,\dots,k\}^{\mathbb{Z}^2}$ such that if $\sigma, \tau \colon \widehat{X} \to \widehat{X}$ denote the horizontal and vertical shifts (respectively) and if ν is the unique invariant Borel probability on $\widehat{X}$ and $\mathcal{B}$ denotes the Borel σ-algera, then $(\widehat{X}, \mathcal{B}, \nu, \sigma, \tau)$ is a topological model for $(X, \mathcal{X}, \mu, S, T)$.*

We note that in [15], the proof given shows that $\widehat{X} \subset \{1,\dots,k+1\}^{\mathbb{Z}^2}$ and the result that the shift alphabet can be taken to have only k letters is stated without proof. However, the size of the alphabet is not relevant for our purposes, other than the fact that it is a finite number.

The subshift $\widehat{X} \subset \{1,\dots,k\}^{\mathbb{Z}^2}$ in the conclusion of Theorem 1.3 is not uniquely defined, and so we make the following definition:

Definition 1.4. Let $(X, \mathcal{X}, \mu, S, T)$ be an ergodic $\mathbb{Z}^2$-system. A minimal, uniquely ergodic $\mathbb{Z}^2$-subshift that is measure theoretically isomorphic to $(X, \mathcal{X}, \mu, S, T)$ is called a *Jewett-Krieger model for* $(X, \mathcal{X}, \mu, S, T)$.

Theorem 1.3 guarantees that any free ergodic $\mathbb{Z}^2$ system of finite entropy has a Jewett-Krieger model. However the definition is still valid for non-free, ergodic $\mathbb{Z}^2$ systems; the only difference is that Rosenthal's Theorem no longer guarantees that such a model exists. For the case of interest to us, we show (in the proof of Theorem 1.9) that if μ is $\langle S,T\rangle$-ergodic, then either μ is atomic or the action of $\langle S,T\rangle$ is free. This motives us to make the following observation: a finite, ergodic $\mathbb{Z}^2$-system cannot be free, but it has a Jewett-Krieger model in a trivial way, obtained by partitioning the system into individual points.

Using this language, we can rephrase Furstenberg's Conjecture and the Rudolph-Johnson Theorem as equivalent statements about Jewett-Krieger models. Fix the transformations $S, T \colon [0,1) \to [0,1)$ to be the maps $Sx := px \pmod 1$ and $Tx := qx \pmod 1$, where $p, q \geq 1$ are multiplicatively independent integers. By the natural extension, we mean the invertible cover (see Section 2.1).

Conjecture 1.5 (Symbolic Furstenberg Conjecture)**.** *Let μ be a Borel probability measure on $[0,1)$ with Borel σ-algebra $\mathcal{B}$ that is invariant under both S and T and ergodic for the joint action. If $\widehat{X} \subset \{0,1\}^{\mathbb{Z}^2}$ is a Jewett-Krieger model for the natural extension of $([0,1), \mathcal{B}, \mu, S, T)$, then either $\widehat{X}$ is finite or μ is Lebesgue measure.*

Theorem 1.6 (Symbolic Rudolph-Johnson Theorem)**.** *Let μ be a Borel probability measure on $[0,1)$ with Borel σ-algebra $\mathcal{B}$ which is invariant under both S and T and is ergodic for the joint action. Let $\widehat{X} \subset \{0,1\}^{\mathbb{Z}^2}$ be a Jewett-Krieger model for the natural extension of $([0,1), \mathcal{B}, \mu, S, T)$ and let $\sigma, \tau \colon \widehat{X} \to \widehat{X}$ denote the horizontal and vertical shifts (respectively). If either $h_\nu(\sigma) > 0$ or $h_\nu(\tau) > 0$, then μ is Lebesgue measure.*

Proof. An isomorphism of the $\mathbb{Z}^2$-systems $(X, \mathcal{X}, \mu, S, T)$ and $(\widehat{X}, \mathcal{B}, \nu, \sigma, \tau)$ restricts to an isomorphism of the $\mathbb{Z}$-systems $(X, \mathcal{X}, \mu, S)$ and $(\widehat{X}, \mathcal{B}, \nu, \sigma)$, and so $h_\mu(S) = h_\nu(\sigma)$. Similarly $h_\mu(T) = h_\nu(\tau)$. The statement then follows immediately from the Rudolph-Johnson Theorem. □

1.4. **Combinatorial rephrasing of measure theoretic entropy.** The appeal of Theorem 1.6 is that the hypothesis that $h_\nu(\sigma) > 0$ (or equivalently that $h_\nu(\tau) > 0$) can be phrased purely as a combinatorial statement about the frequency with which words in the language of $\widehat{X}$ occur in larger words in the language of $\widehat{X}$. To explain this, we start with some definitions.

If $X \subset \mathcal{A}^{\mathbb{Z}}$ is a subshift over the finite alphabet $\mathcal{A}$, we write $x = (x(n) \colon n \in \mathbb{Z})$. A *word* is a defined to be a finite sequence of symbols contained consecutively in some x and we let $|w|$ denote the number of symbols in w (it may be finite or infinite). A word w is a *subword* of a word u if the symbols in the word w occur somewhere in u as consecutive symbols. The *language* $\mathcal{L} = \mathcal{L}(X)$ of X is defined to be the collection of all finite subwords that arise in elements of X. If $w \in \mathcal{L}(X)$, let $[w]$ denote the *cylinder set* it determines, meaning that

$$[w] = \{u \in \mathcal{L} \colon u(n) = w(n) \text{ for } 1 \le n \le |w|\}.$$

These definitions naturally generalize to a two dimensional subshift $X \subset \mathcal{A}^{\mathbb{Z}^2}$, and for $x \in \mathcal{A}^{\mathbb{Z}^2}$ we write $x = (x(\vec{u}) \colon \vec{u} \in \mathbb{Z}^2)$. A *word* is a finite, two dimensional configuration that is convex and connected (as a subset of $\mathbb{Z}^2$), and a *subword* is a configuration contained in another word. If $F \subset \mathbb{Z}^2$ is finite and $\beta \in \mathcal{A}^F$, then the *cylinder set of shape F determined by β* is defined to be the set

$$[F; \beta] := \{x \in \mathcal{A}^{\mathbb{Z}^2} \colon x(\vec{u}) = \beta(\vec{u}) \text{ for all } \vec{u} \in F\}.$$

Lemma 1.7. *Let $(\widehat{X}, \mathcal{B}, \nu, \sigma, \tau)$ be a strictly ergodic $\mathbb{Z}^2$-subshift. Let w be a $(2n+1)\times(2n+1)$ word in the language of $\widehat{X}$ and let $[w]$ denote the cylinder set determined by placing the word w centered at $(0,0)$. Let $u_1, u_2, u_3, \dots$ be words in the language of $\widehat{X}$ such that u_i is a square of size $(2n+2i+1) \times (2n+2i+1)$. If $N(w, u_i)$ denotes the number of times w occurs as a subword of u_i, then*

$$\nu[w] = \lim_{i\to\infty} N(w, u_i)/(2i+1)^2.$$

Proof. By unique ergodicity, the Birkhoff averages of a continuous function converge uniformly to the integral of the function. In particular, this applies to the continuous function $1_{[w]}$, so the limit exists and is independent of the sequence $\{u_i\}_{i=1}^{\infty}$. □

For $m, n \in \mathbb{N}$, let $\mathcal{P}(m,n)$ be the partition of $\widehat{X}$ according to cylinder sets of shape $[0, m-1] \times [-n+1, n-1]$. Observe that (recall that σ, as defined in (1), denotes the left shift)

$$\mathcal{P}(m,n) = \bigvee_{i=0}^{k} \sigma^{-i}\mathcal{P}(1,n)$$

and that $\bigvee_{i=-k}^{k} \sigma^i \mathcal{P}(1,n)$ is the partition of $\widehat{X}$ into symmetric $(2m+1)\times(2n+1)$-cylinders centered at the origin. Therefore, $\{\mathcal{P}(1,n)\}_{n=1}^{\infty}$ is a sufficient (in the sense of Definition 4.3.11 in [7]) family of partitions to generate the Borel σ-algebra of the system $(\widehat{X}, \mathcal{B}, \nu, \sigma)$, where we view this as a $\mathbb{Z}$-system with respect to the horizontal shift σ. Let $h_\nu(\sigma, \mathcal{Q})$ denote the measure theoretic entropy of the system $(\widehat{X}, \mathcal{B}, \nu, \sigma)$ with respect to the partition $\mathcal{Q}$ and let $h_\nu(\sigma)$ denote the measure theoretic entropy of the system. It follows that

$$\begin{aligned} h_\nu(\sigma) &= \sup_n h_\nu(\sigma, \mathcal{P}(1,n)) \\ &= \lim_{n\to\infty} h_\nu\left(\sigma, \mathcal{P}(1,n)\right) \\ &= -\lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{m}\sum_{w\in\mathcal{P}(m,n)} \nu[w]\log\nu[w] \\ &= -\lim_{n\to\infty}\lim_{m\to\infty}\lim_{i\to\infty}\frac{1}{m}\sum_{w\in\mathcal{P}(m,n)} \frac{N(w,u_i)}{(2i+1)^2}\cdot\log\frac{N(w,u_i)}{(2i+1)^2} \end{aligned}$$

by Lemma 1.7. In other words, the Rudolph-Johnson Theorem is equivalent to:

Theorem 1.8 (Combinatorial Rudolph-Johnson Theorem)**.** *Let μ be a Borel probability measure on $[0,1)$ with Borel σ-algebra $\mathcal{B}$ and assume that μ is invariant under both S and T, and ergodic for the joint action. Let $\widehat{X}$ be a Jewett-Krieger model for the natural extension of $([0,1), \mathcal{B}, \mu, S, T)$ and without loss suppose the horizontal shift on $\widehat{X}$ is intertwined with S under this isomoprhism. If*

$$-\lim_{n\to\infty}\lim_{m\to\infty}\lim_{i\to\infty}\frac{1}{m}\sum_{w\in\mathcal{P}(m,n)} \frac{N(w,u_i)}{(2i+1)^2}\cdot\log\frac{N(w,u_i)}{(2i+1)^2} > 0,$$

then the value of this limit is $\log p$ and μ is Lebesgue measure.

1.5. **Complexity of subshifts.** If $X \subset \mathcal{A}^{\mathbb{Z}^2}$ is a nonempty subshift, then its *complexity function* is the function $P_X \colon \{\text{finite subsets of } \mathbb{Z}^2\} \to \mathbb{N}$ given by

$$P_X(F) := \left|\{\beta \in \mathcal{A}^F \colon [F;\beta] \cap X \neq \emptyset\}\right|.$$

Let $R_n := \{(i,j) \in \mathbb{Z}^2 \colon 1 \le i, j \le n\}$ denote the $n \times n$ rectangle in $\mathbb{Z}^2$. A standard notion of the complexity of a subshift $X \subset \mathcal{A}^{Z^2}$ is the asymptotic growth rate of $P_X(R_n)$. Observe that $P_X(R_n)$ is bounded (in n) if and only if X is finite. Moreover, $P_X(R_n)$ grows exponentially if and only if (X, σ, τ) has positive topological entropy.

We are now in a position to state our main technical result.

Theorem 1.9. *Let μ be a Borel probability measure on $[0,1)$ with Borel σ-algebra $\mathcal{B}$. Assume that μ is invariant under both S and T and ergodic for the joint action, and let $\widehat{X} \subset \{0,1\}^{\mathbb{Z}^2}$ be a Jewett-Krieger model for the natural extension of $([0,1), \mathcal{B}, \mu, S, T)$. If there exists $n \in \mathbb{N}$ such that $P_{\widehat{X}}(R_n) \le \frac{1}{2}n^2$, then $P_{\widehat{X}}(R_n)$ is bounded (independent of n) and $\widehat{X}$ is finite. In particular, μ is atomic.*

This gives a nontrivial complexity gap for the set of $\langle S,T\rangle$-ergodic probability measures, which is our main result:

Corollary 1.10 (Complexity gap)**.** *Let μ be a Borel probability measure on $[0,1)$ which is invariant under both S and T and ergodic for the joint action, and let $\widehat{X}\subset\{0,1\}^{\mathbb{Z}^2}$ be a Jewett-Krieger model for the natural extension of $([0,1),\mathcal{B},\mu,S,T)$. Then either $P_{\widehat{X}}(R_n)$ is bounded (and μ is atomic) or*

$$\liminf_{n\to\infty}\frac{P_{\widehat{X}}(R_n)}{n^2}\geq\frac{1}{2}.$$

This gap is nontrivial in the following sense: there exist infinite (i.e. not doubly periodic), strictly ergodic $\mathbb{Z}^2$-subshifts whose complexity function grows subquadratically. The statement made by Corollary 1.10 is that any such subshift cannot be a Jewett-Krieger model of any $\langle\times p,\times q\rangle$-ergodic measure on $[0,1)$.

Example 1.11. Let $X\subset\{0,1\}^{\mathbb{Z}}$ be a Sturmian shift (see [10] for the definition). Then X is strictly ergodic and $P_X(n)=n+1$ for all $n\in\mathbb{N}$. Let $Y\subset\{0,1\}^{\mathbb{Z}^2}$ be the subshift whose points are obtained by placing each $x\in X$ along the x-axis in $\mathbb{Z}^2$ and then copying vertically (i.e. each point in Y is vertically constant and its restriction to the x-axis is an element of X). It follows that Y is strictly ergodic and that $P_Y(R_n)=n+1$ for all $n\in\mathbb{N}$. Corollary 1.10 shows that Y is not a Jewett-Krieger model for any $\langle\times p,\times q\rangle$-ergodic measure on $[0,1)$.

1.6. **Remarks on complexity growth.** We conclude our introduction with a few brief remarks on Theorem 1.9 and Corollary 1.10. We show (Lemma 2.1) that any Jewett-Krieger model $\widehat{X}$ for an atomic $\langle S,T\rangle$-ergodic measure is a strictly ergodic $\mathbb{Z}^2$-subshift containing only doubly periodic $\mathbb{Z}^2$-colorings, meaning that there are only finitely many points in $\widehat{X}$. From this, it is easy to deduce that $P_{\widehat{X}}(R_n)$ is bounded independently of n (by the number of points in $\widehat{X}$). Moreover, we show that if $\widehat{X}$ is a Jewett-Krieger model for μ and if $\widehat{X}$ contains only doubly periodic $\mathbb{Z}^2$-colorings, then μ is atomic.

A strategy for proving Theorem 1.9 is therefore to find a nontrivial growth rate of $P_{\widehat{X}}(R_n)$ which implies that $\widehat{X}$ contains only doubly periodic $\mathbb{Z}^2$-colorings. A simple example of such a rate follows from the classical Morse-Hedlund Theorem [10]: if there exists $n\in\mathbb{N}$ such that $P_{\widehat{X}}(R_n)\leq n$, then $\widehat{X}$ contains only doubly periodic $\mathbb{Z}^2$-colorings (see e.g. the proof of Theorem 1.2 in [13]). In fact this bound is sharp: there exist $\mathbb{Z}^2$-colorings that are not doubly periodic and yet satisfy $P_{\widehat{X}}(R_n)=n+1$ for all $n\in\mathbb{N}$. Many other subquadratic growth rates can also be realized by strictly ergodic $\mathbb{Z}^2$-subshifts that do not contain doubly periodic points (see, for example, [12]). Therefore, a weak version of Theorem 1.9 that replaces the assumption that there exists $n\in\mathbb{N}$ such that $P_{\widehat{X}}(R_n)\leq\frac{1}{2}\cdot n^2$ with the stronger assumption that there exists $n\in\mathbb{N}$ such that $P_{\widehat{X}}(R_n)\leq n$, follows from the Morse-Hedlund Theorem. However, this weak theorem relies on the fact that there are simply no strictly ergodic $\mathbb{Z}^2$-subshifts for which $P_{\widehat{X}}(R_n)$ is unbounded but for which $P_{\widehat{X}}(R_n)\leq n$ (for some n). The complexity gap provided by this weak theorem is therefore trivial in the sense that there are no strictly ergodic $\mathbb{Z}^2$-subshifts whose complexity function lies in this gap.

On the other hand, there do exist strictly ergodic $\mathbb{Z}^2$-subshifts with unbounded complexity and such that $P_{\widehat{X}}(R_n)<\frac{1}{2}\cdot n^2$. This is the interest in Theorem 1.9 and Corollary 1.10. The content of the theorem is that although such $\mathbb{Z}^2$-systems exist, they can not be Jewett-Krieger models of $\langle S,T\rangle$-ergodic measures on $[0,1)$. This is analogous to Theorem 1.8, which says that although there are strictly ergodic $\mathbb{Z}^2$-subshifts that have small but positive entropy, they are not Jewett-Krieger models of $\langle S,T\rangle$-ergodic measures on $[0,1)$. Moreover, analogous to the hypothesis of Corollary 1.10 which relies on the growth rate of $P_{\widehat{X}}(\cdot)$, the hypothesis of Theorem 1.8 is a condition on the growth rate of the relative complexity function $N(\cdot,\cdot)$ of Lemma 1.7, with respect to the action of the horizontal shift (a similar statement holds for the vertical shift).

2. Proof of Theorem 1.9

Throughout this section, we assume that $p,q\geq 2$ are multiplicatively independent integers and that μ is a Borel probability measure on $[0,1)$ which is invariant under both

$$\begin{aligned} Sx &:= px \ (\mathrm{mod}\ 1); \\ Tx &:= qx \ (\mathrm{mod}\ 1) \end{aligned}$$

and is ergodic with respect to the joint action $\langle S,T\rangle$. Let $\mathcal{B}$ denote the associated Borel σ-algebra on $[0,1)$

2.1. **The natural extension.** Let X be the natural extension of the $\mathbb{N}^2$-system $([0,1),\mathcal{B},\mu,S,T)$. Specifically (following [14]), let

$$X := \left\{ y\in[0,1)^{\mathbb{Z}^2} : y(i+1,j)=Sy(i,j) \text{ and } y(i,j+1)=Ty(i,j) \text{ for all } i,j\in\mathbb{Z}\right\},$$

and for $(i,j)\in\mathbb{Z}^2$ let $\pi_{(i,j)}\colon X\to[0,1)$ be the map $\pi_{(i,j)}(y)=y(i,j)$. Define a countably additive measure μ_X on the σ-algebra

$$\bigcup_{i=0}^{\infty}\pi^{-1}_{(-i,-i)}\mathcal{B}$$

by setting $\mu_X(\pi^{-1}_{(-i,-i)}A):=\mu(A)$. Let $\mathcal{X}$ be the completion of this σ-algebra with respect to μ_X. Let $S_X,T_X\colon X\to X$ be the left shift and the down shift, respectively. Thus $\pi_{(0,0)}$ defines a measure theoretic factor map from $(X,\mathcal{X},\mu_X,S_X,T_X)$ to $([0,1),\mathcal{B},\mu,S,T)$. Moreover, μ_X is ergodic if and only if μ is ergodic. By construction, $h_\mu(S)=h_{\mu_X}(S_X)$, $h_\mu(T)=h_{\mu_X}(T_X)$, and $h_\mu(\langle S,T\rangle)=h_{\mu_X}(\langle S_X,T_X\rangle)$.

The advantage of working with $(X,\mathcal{X},\mu_X,S_X,T_X)$ instead of the original system is that the natural extension is an ergodic $\mathbb{Z}^2$-system.

2.2. **Jewett-Krieger models and periodicity.** If the two dimensional entropy of a system is positive, then the entropy of every one dimensional subsystem is infinite (for a proof, see, for example, [17]). In our setting, since $h_\mu(S)\leq h_{\mathrm{top}}(S)=\log(p)$ (and $h_\mu(T)\leq h_{\mathrm{top}}(T)=\log(q)$), it follows that the measure theoretic entropy of the joint action generated by $\langle S,T\rangle$ on $[0,1)$ with respect to μ is also zero. It follows that the measure theoretic entropy with respect to μ_X of the joint action on X generated by $\langle S_X,T_X\rangle$ is zero. Therefore, by Theorem 1.3, there exists a strictly ergodic subshift $\widehat{X}\subset\{0,1\}^{\mathbb{Z}^2}$ such that $(X,\mathcal{X},\mu_X,S_X,T_X)$ is measure theoretically isomorphic to $(\widehat{X},\widehat{\mathcal{X}},\nu,\sigma,\tau)$, where $\widehat{\mathcal{X}}$ is the Borel σ-algebra on $\widehat{X}$, $\sigma,\tau\colon\widehat{X}\to\widehat{X}$ denote the left shift and down shift (respectively), and ν is the unique $\langle\sigma,\tau\rangle$-invariant Borel probability measure. Note that the choice of $\widehat{X}$ is not necessarily unique.

Lemma 2.1. *If $(X, \mathcal{X}, \mu_X, S_X, T_X)$ is an atomic system, then any Jewett-Krieger model $(\widehat{X}, \widehat{\mathcal{X}}, \nu, \sigma, \tau)$ for $(X, \mathcal{X}, \mu_X, S_X, T_X)$ is finite.*

Proof. Let $\pi\colon (\widehat{X}, \widehat{\mathcal{X}}, \nu, \sigma, \tau) \to (X, \mathcal{X}, \mu_X, S_X, T_X)$ be an isomorphism and let $x \in X$ be an atom. Then there exist full measure sets $\widehat{X}_1 \subset \widehat{X}$ and $X_1 \subset X$ such that $\pi\colon \widehat{X}_1 \to X_1$ is a bijection which interwines the $\mathbb{Z}^2$ actions. Every atom in X is contained in X_1, and if $x \in X_1$ is an atom then there exists unique $y \in \widehat{X}_1$ such that $\pi(y) = x$. It follows that $\nu(\{y\}) = \mu_X(\{x\}) > 0$ and so y is an atom in $\widehat{X}$. By the Poincaré Recurrence Theorem, there exists $(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $S_X^i T_X^j y = y$. Let $V_y := \{(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}\colon S_X^i T_X^j y = y\}$ be the (nonempty) set of nontrivial period vectors for y. If $\dim(\mathrm{Span}(V_y)) = 1$, then

$$\lim_{N\to\infty} \frac{1}{(2N+1)^2} \sum_{-N\le i,j\le N} 1_{\{y\}}(S_X^i T_X^j y) = 0 < \nu(\{y\}),$$

which contradicts the pointwise ergodic theorem. Therefore $\dim(\mathrm{Span}(V_y)) = 2$ and $y \in \mathcal{A}^{\mathbb{Z}^2}$ is doubly periodic. Moreover, for ν-a.e. $z \in \widehat{X}$ we have $S_X^i T_X^j z = y$ for some $(i, j) \in \mathbb{Z}^2$ and so z is also doubly periodic (with periods equal to those of y). Thus there are only finitely many points $z \in \widehat{X}$. □

Since $\widehat{X}$ is minimal, and hence transitive, we can use the following tool for studying the dynamics of $(X, \mathcal{X}, \mu_X, S_X, T_X)$:

Theorem 2.2 (Cyr & Kra [2])**.** *If (X, σ, τ) is a transitive $\mathbb{Z}^2$-subshift and there exist $n, k \in \mathbb{N}$ such that $P_X(n, k) \le nk/2$, then there exists $(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $\sigma^i\tau^j x = x$ for all $x \in X$.*

Lemma 2.3. *If there exists $(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0\}$ such that $\sigma^i\tau^j x = x$ for every $x \in \widehat{X}$, then $S_X^i T_X^j x = x$ for μ-almost every $x \in X$.*

Proof. Let $\psi\colon \widehat{X} \to X$ be an isomorphism. Thus there exist $\widehat{X}_1 \subset \widehat{X}$ and $X_1 \subset X$ such that $\nu(\widehat{X}_1) = \mu_X(X_1) = 1$, $\psi\colon \widehat{X}_1 \to X_1$ is a bi-measurable bijection, $\psi_*\nu = \mu_X$, $\psi \circ \sigma = S_X \circ \psi$, and $\psi \circ \tau = T_X \circ \psi$. Let $E = \{x \in X_1\colon S_X^i T_X^j x \ne x\}$. Since $\psi^{-1}(E) = \{y \in \widehat{X}_1\colon \sigma^i\tau^j y \ne y\}$, it follows that $\mu_X(E) = \nu(\psi^{-1}(E)) = 0$. □

Theorem 2.4. *If there exist $n, k \in \mathbb{N}$ such that $P_{\widehat{X}}(n, k) \le nk/2$, then μ is atomic. Moreover, if $\widehat{Y}$ is any other Jewett-Krieger model for $([0, 1), \mathcal{B}, \mu, S, T)$, then $P_{\widehat{Y}}(n, k)$ is bounded independent of $n, k \in \mathbb{N}$.*

Proof. Combining Theorem 2.2 and Lemma 2.3, there exist $(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0\}$ such that $S_X^i T_X^j x = x$ for μ_X-a.e. $x \in X$. Therefore $(S_X^i T_X^j x)(0, 0) = x(0, 0)$ for μ_X-a.e. $x \in X$. It is immediate that we also have $(S_X^{-i} T_X^{-j} x)(0, 0) = x(0, 0)$ for μ_X-a.e. $x \in X$. So there are two cases to consider, depending on the the sign of $i \cdot j$.

Case 1. Suppose $i \cdot j \ge 0$. Then, replacing by $-i$ and $-j$ if necessary, we can assume that both i and j are nonnegative. Set $E := \{y \in [0, 1)\colon S^iT^j y \ne y\}$ and let $y \in E$. Then if $x \in \pi^{-1}(y)$, we have that $S_X^i T_X^j x \ne x$. Thus $\mu(E) = \mu_X(\pi^{-1}(E)) = 0$ and so $S^iT^j y = y$ for μ-a.e. $y \in [0, 1)$.

Now observe that $S^iT^j y = y$ is equivalent to the statement that

$$p^iq^j y = y \pmod 1,$$

which only has finitely many solutions in the interval $[0,1)$. Therefore, μ is supported on a finite set. Since μ is $\langle S,T\rangle$-invariant, this set must be S- and T-invariant. Therefore there exist $a,b\in\mathbb{N}$ such that S^a and T^b are both equal to the identity μ-almost everywhere.

Case 2. Suppose $i\cdot j<0$. Again, replacing by $-i$ and $-j$ if necessary, we can assume that $i<0$ and $j>0$. Now set $E:=\{y\in[0,1)\colon S^{|i|}y\neq T^jy\}$. Thus if $y\in E$ and $x\in\pi^{-1}(y)$, then $x(-i,j)\neq x(0,0)=y$ as $S^{|i|}(x(-i,j))=x(0,j)=T^j(x(0,0))$ by construction. Therefore $S^iT^jx\neq x$ and so $\mu(E)=\mu_X(\pi^{-1}(E))=0$. It follows that $S^{|i|}y=T^jy$ for μ-a.e. $y\in[0,1)$.

Finally observe that $S^{|i|}y=T^jy$ is equivalent to

$$p^{|i|}y=q^jy \pmod 1.$$

As p and q are multiplicatively independent, there are only finitely many solutions in the interval $[0,1)$. Therefore, again, μ is supported on a finite set and there exist $a,b\in\mathbb{N}$ such that S^a and T^b are both equal to the identity μ-almost everywhere.

This establishes the first claim of the theorem. By Lemma 2.1, any Jewett-Krieger model of an atomic system is finite, and the second statement follows. □

We use this to complete the proof of Theorem 1.9:

Proof of Theorem 1.9. Let μ be a Borel probability measure on $[0,1)$ that is $\langle S,T\rangle$-ergodic. If this two dimensional action is not free, arguing as in the proof of Theorem 2.4 that μ is an atomic measure, we are done. Thus we can assume that the action is free, and similarly the action for the natural extension is also free.

Let $(\widehat{X},\widehat{\mathcal{X}},\nu,\sigma,\tau)$ be a Jewett-Krieger model for the natural extension of the system $([0,1),\mathcal{B},\mu,S,T)$. If there is no such model satisfying the additional property that there exist $n,k\in\mathbb{N}$ satisfying $P_{\widehat{X}}(n,k)\le nk/2$, then the conclusion of the Theorem holds vacuously. Thus it suffices to assume that there exists a Jewett-Krieger model $(\widehat{X},\widehat{\mathcal{X}},\nu,\sigma,\tau)$ with the property that there exist $n,k\in\mathbb{N}$ satisfying $P_{\widehat{X}}(n,k)\le nk/2$. By Theorem 2.4, $([0,1),\mathcal{B},\mu,S,T)$ is atomic. □

3. Higher dimensions

Theorem 1.9 shows that if μ is any nonatomic $\times p$, $\times q$ ergodic measure then the natural extension of $([0,1),\mathcal{X},\mu,S,T)$ cannot be measurably isomorphic to any $\mathbb{Z}^2$-subshift of whose complexity function satisfies $P_X(n,n)=o(n^2)$. It is natural to ask whether this result can be generalized to higher dimensions. In particular, if $p_1,\dots,p_d$ are a multiplicatively independent set of integers and μ is a nonatomic $\times p_1,\dots,\times p_d$ ergodic measure, we can ask if the natural extension of $(X,\mathcal{X},\mu,\times p_1,\dots,\times p_d)$ could have a topological model whose complexity function is $o(n^d)$.

The same method used in the two dimensional case suggests a path to proving this result. If one could show that any free, strictly ergodic $\mathbb{Z}^d$-subshift whose complexity function is $o(n^d)$ is periodic, then it would follow that no such topological model for μ exists. However, the analog of Theorem 2.2 in dimension $d>2$ is false. Julien Cassaigne [1] has shown that for $d>2$, there exists a minimal $\mathbb{Z}^d$-subshift X whose elements are not periodic in any direction, and is such that for any $\varepsilon>0$ we have $P_X(n,n,\dots,n)=o(n^{2+\varepsilon})$. On the other hand, the authors have recently shown [3] that the analog of Theorem 2.2 does hold for dimension $d>2$ if a certain expansiveness assumption is imposed on the subshift.

If $Y \subset \mathcal{A}^{\mathbb{Z}^d}$ is a subshift, then we say that the x-axis in $\mathbb{Z}^d$ is *strongly expansive* if whenever $x, y \in X$ have the same restriction to the x-axis, we have $x = y$. In this case, if $X \subset \mathcal{A}^{\mathbb{Z}}$ is the subshift obtained by restricting elements of Y to the x-axis, then there exist homeomorphisms $\tau_1, \ldots, \tau_{d-1} \colon X \to X$ which commute pairwise and with the shift σ and are such that for any $y \in Y$ we have $y(i_1, i_2, \ldots, i_d) = \left(\tau_1^{i_1}\tau_2^{i_2}\cdots\tau_{d-1}^{i_{d-1}}\sigma^{i_d}\pi_X(y)\right)(0)$ for all $i_1, \ldots, i_d \in \mathbb{Z}^d$, where $\pi_X(y)$ denotes the restriction of y to the x-axis. In previous work, we have shown that:

Theorem 3.1 (Cyr & Kra [3]). *Let $X \subset \mathcal{A}^{\mathbb{Z}}$ be a minimal subshift and let $\tau_1, \ldots, \tau_{d-1} \colon X \to X$ be homeomorphisms of X that commute with the shift σ. If $\langle \sigma, \tau_1, \ldots, \tau_{d-1}\rangle \cong \mathbb{Z}^d$, then $\liminf_{n\to\infty} P_X(n)/n^d > 0$.*

With some additional effort, the same result can be shown if the assumption that (X, σ) is minimal (as a $\mathbb{Z}$-system) is relaxed to only require that $(X, \sigma, \tau_1, \ldots, \tau_d)$ is minimal (as a $\mathbb{Z}^d$-system). Thus, the only obstruction to generalizing Theorem 1.9 to the higher dimensional setting is the following:

Conjecture 3.2. *For every nonatomic Borel probability μ on $[0, 1)$ which is ergodic for the joint action of $\times p_1, \ldots, \times p_d$, there is a strongly expansive, minimal topological model for $(X, \mathcal{X}, \mu, \times p_1, \ldots, \times p_d)$.*

If this conjecture holds, then if follows that any such system is measurably isomorphic to a subshift whose complexity function grows on the order of n^d.

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Bucknell University, Lewisburg, PA 17837 USA
E-mail address: van.cyr@bucknell.edu

Northwestern University, Evanston, IL 60208 USA
E-mail address: kra@math.northwestern.edu