THE SPACETIME OF A SHIFT AUTOMORPHISM

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Abstract. The automorphism group of a one dimensional shift space over a finite alphabet exhibits different types of behavior: for a large class with positive entropy, it contains a rich collection of subgroups, while for many shifts of zero entropy, there are strong constraints on the automorphism group. We view this from a different perspective, considering a single automorphism and studying the naturally associated two dimensional shift system. In particular, we describe the relation between nonexpansive subspaces in this two dimensional system and dynamical properties of an automorphism of the shift.

Contents

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1. Introduction

Suppose Σ is a finite alphabet and $X \subset \Sigma^{\mathbb{Z}}$ is a closed set that is invariant under the left shift $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$. The collection of automorphisms group $\mathrm{Aut}(X, \sigma)$, consisting of all homeomorphisms $\phi\colon X \to X$ that commute with σ, forms a group (under composition). A useful approach to understanding a countable group G is knowing if it has subgroups which are isomorphic to (or are homomorphic images of) simpler groups which are relatively well understood, such as matrix groups, and in particular, lattices in classical Lie groups. While the automorphism group of a shift is necessarily countable (as an immediate corollary of the Curtis-Hedlund-Lyndon Theorem [11], any automorphism $\phi\colon X \to X$ is given by a block code), there are numerous results in the literature showing that the automorphism group of the full shift, and more generally any mixing shift of finite type, contains isomorphic copies of many groups: this collections includes, for example, any finite group, the direct sum of countably many copies of $\mathbb{Z}$, the free group on any finite number of generators, and the fundamental group of any 2-manifold (see [11, 4, 13]). In light of these results, it is natural to ask if there is any finitely generated (or even countable) group which fails to embed in any such automorphism group, meaning any group of the form $\mathrm{Aut}(X, \sigma)$. A partial answer is given in [1], where it is shown that if (X, σ) is a subshift of finite type then any group that embeds in the automorphism group must be residually finite. At the other end of the complexity spectrum for (X, σ), there has been recent work showing that $\mathrm{Aut}(X, \sigma)$ is significantly more tame for a shift with very low complexity (see for example [6, 7, 9]).

Instead of viewing the entire group, we focus on the structure inherent in a single automorphism $\phi \in \mathrm{Aut}(X, \sigma)$, as studied for example in [11, 1, 13, 14]. Given an automorphism ϕ, there is an obvious way to associate a $\mathbb{Z}^2$-shift action, which we call the spacetime of ϕ (in a slightly different setting, this is called the complete history by Milnor [17] and is referred to as the spacetime diagram in the cellular automata literature). We make use of a particular subset of the spacetime, dubbed the light cone, that is closely related to the notion of causal cone discussed in [17]. We show that the light cone gives a characterization of a well studied structural feature of a $\mathbb{Z}^2$-shift, namely the set of nonexpansive subspaces (see [3] and [12]). In particular, in §4 we show that the edges of a light cone for ϕ are always nonexpansive subspaces in its spacetime (the precise statement is given in Theorem 4.4).

We also provide a partial converse to this result: for many $\mathbb{Z}^2$-systems with nonexpansive subspace L, the system is isomorphic to the space time of an automorphism ϕ by an isomorphism which carries L to an edge of the light cone of ϕ.

We then use these structural results to describe obstructions to embedding in the automorphism group of a shift. An important concept in the study of lattices is the idea of a distortion element, meaning an element whose powers have sublinear growth of their minimal word length in some (and hence any) set of generators. In §5, we introduce a notion of range distortion for automorphisms, meaning that the associated block codes of iterates of the automorphism grow sublinearly. An immediate observation is that if an automorphism is distorted in $\mathrm{Aut}(X)$ (in the group sense), then it is also range distorted. We also introduce a measure of non-distortion called the asymptotic spread $A(\phi)$ of an automorphism ϕ and show that the topological entropies of ϕ and σ satisfy the inequality

$$h_{\mathrm{top}}(\phi) \leq A(\phi) h_{\mathrm{top}}(\sigma).$$

This recovers an inequality of Tisseur [20]; his context is more restrictive, covering the full shift endowed with the uniform measure. We do not appeal to measure theoretic entropy and our statement applies to a wider class of shifts.

This inequality proves to be useful in providing obstructions to various groups embedding in the automorphism group. These ideas are further explored in [8].

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2. Background

2.1. **Shift systems and automorphisms.** We assume throughout that Σ is a finite set (which we call the *alphabet*) endowed with the discrete topology and endow $\Sigma^{\mathbb{Z}}$ with the product topology. For $x \in \Sigma^{\mathbb{Z}}$, we write $x[n] \in \Sigma$ for the value of x at $n \in \mathbb{Z}$.

The *left shift* $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is defined by $(\sigma x)[n] = x[n+1]$, and is a homeomorphism from $\Sigma^{\mathbb{Z}}$ to itself. We say that (X, σ) is a *subshift*, or just a *shift* when the context is clear, if $X \subset \Sigma^{\mathbb{Z}}$ is a closed set that is invariant under the left shift $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$.

Standing assumption: Throughout this article, (X, σ) denotes a shift system and we assume that the alphabet Σ of X is finite and that the shift (X, σ) is infinite, meaning that $|X| = \infty$.

An *automorphism* of the shift (X, σ) is a homeomorphism $\phi\colon X \to X$ such that $\phi \circ \sigma = \sigma \circ \phi$. The group of all automorphisms of (X, σ) is denoted $\mathrm{Aut}(X, \sigma)$, or simply $\mathrm{Aut}(X)$ when σ is clear from the context.

For an interval $[n, n+1, \ldots, n+k-1] \in \mathbb{Z}$ and $x \in X$, we let $x[n, \ldots, n+k-1]$ denote the element a of Σ^k with $a_j = x[n+j]$ for $j = 0, 1, \ldots, k-1$. Define the *words* $\mathcal{L}_k(X)$ *of length* k *in* X to be the collection of all $[a_1, \ldots, a_k] \in \Sigma^k$ such that there exist $x \in X$ and $m \in \mathbb{Z}$ with $x[m+i] = a_i$ for $1 \leq i \leq k$. The length of a word $w \in \mathcal{L}(X)$ is denoted by $|w|$. The language $\mathcal{L}(X) = \bigcup_{k=1}^{\infty} \mathcal{L}_k(X)$ is defined to be the collection of all finite words.

The *complexity* of (X, σ) is the function $P_X\colon \mathbb{N} \to \mathbb{N}$ that counts the number of words of length n in the language of X. Thus

$$P_X(n) = \big|\mathcal{L}_n(X)\big|.$$

The exponential growth rate of the complexity is the *topological entropy* h_{top} of the shift σ. Thus

$$h_{\mathrm{top}}(\sigma) = \lim_{n\to\infty} \frac{\log(P_X(n))}{n}.$$

This is equivalent to the usual definition of topological entropy using (n, ε)-separated sets (see, for example [16]).

A map $\phi\colon X \to X$ is a *sliding block code* if there exists $R \in \mathbb{N}$ such that for any $x, y \in X$ with $x[i] = y[i]$ for $-R \leq i \leq R$, we have that $\phi(x)[0] = \phi(y)[0]$. The least R such that this holds is called the *range* of ϕ.

By the Curtis-Hedlund-Lyndon Theorem [11], any automorphism $\phi\colon X \to X$ of a shift (X, σ) is a sliding block code. In particular, $\mathrm{Aut}(X)$ is always countable.

Definition 2.1. Suppose (X, σ) and (X', σ') are shifts and $\phi \in \mathrm{Aut}(X, \sigma)$ and $\phi' \in \mathrm{Aut}(X', \sigma')$ are automorphisms. We say that ϕ and ϕ' are *conjugate automorphisms* if there is a homeomorphism $h\colon X \to X'$ such that

$$h \circ \sigma = \sigma' \circ h \text{ and } h \circ \phi = \phi' \circ h.$$

A homeomorphism h satisfying these properties is a sliding block code. If ϕ and ϕ' both lie in $\mathrm{Aut}(X, \sigma)$, then ϕ and ϕ' are conjugate if and only if they are conjugate as elements of the group $\mathrm{Aut}(X, \sigma)$.

A shift X is *irreducible* if for all words $u, v \in \mathcal{L}(X)$, there exists $w \in \mathcal{L}(X)$ such that $uwv \in \mathcal{L}(X)$.

Definition 2.2. A shift (X, σ) is a *subshift of finite type* provided it is defined by a finite set of excluded words, $\mathcal{F} \subset \mathcal{L}(\Sigma^{\mathbb{Z}})$. In other words, $x \in X$ if and only if there are no $n \in \mathbb{Z}$ and $k > 0$ such that $x[n, \dots, n + k] \in \mathcal{F}$.

We make use of the following proposition due to Bowen [2]. A proof can be found in [16, Theorem 2.1.8].

Proposition 2.3. *A shift (X, σ) is a shift of finite type if and only if there exists $n_0 \geq 0$ such that whenever $uw, wv \in \mathcal{L}(X)$ and $|w| \geq n_0$, then also $uwv \in \mathcal{L}(X)$.*

2.2. **Higher dimensions.** More generally, we can consider a multidimensional shift $X \subset \Sigma^{\mathbb{Z}^d}$ for some $d \geq 1$, where X is a closed set (with respect to the product topology) that is invariant under the $\mathbb{Z}^d$ action $(T^u x)(v) = x(u + v)$ for $u \in \mathbb{Z}^d$. We refer to $\eta \in X$ as an *X-coloring of $\mathbb{Z}^d$*.

We note that we have made a slight abuse of notation in passing to the multidimensional setting by denoting the entries of an element $x \in X$ by $x(u)$ (where $u \in \mathbb{Z}^d$), rather than $x[u]$ as we did for a one dimensional shift. This is done to avoid confusion with interval notation, as we frequently restrict ourselves to the two dimensional case, writing $x(i, j)$ rather than the possibly confusing $x[i, j]$.

Definition 2.4. Suppose $X \subset \mathbb{Z}^d$ is a topological $\mathbb{Z}^d$-system, endowed with the natural $\mathbb{Z}^d$-action by translations. If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and $\alpha\colon \mathcal{S} \to \Sigma$, define the *cylinder set*

$$[\mathcal{S}, \alpha] := \{\eta \in X \colon \text{the restriction of } \eta \text{ to } \mathcal{S} \text{ is } \alpha\}.$$

The set of all cylinder sets forms a basis for the topology of X. The *complexity function* for X is the map $P_X\colon \{\text{finite subsets of } \mathbb{Z}^d\} \to \mathbb{N}$ given by

$$P_X(\mathcal{S}) := \left|\{\alpha \in \Sigma^{\mathcal{S}} \colon [\mathcal{S}, \alpha] \neq \emptyset\}\right|$$

which counts the number of colorings of $\mathcal{S}$ which are restrictions of elements of X. If $\alpha\colon \mathcal{S} \to \Sigma$, is the restriction of an element of X we say it *extends uniquely* to an

X-coloring if there is exactly one legal $\eta \in X$ whose restriction to $\mathcal{S}$ is α. Similarly, if $\mathcal{S} \subset \mathcal{T} \subset \mathbb{Z}^d$ and if $\alpha \colon \mathcal{S} \to \Sigma$ is such that $[\alpha, \mathcal{S}] \neq \emptyset$, then we say α *extends uniquely* to an X-coloring of $\mathcal{T}$ if there is a unique $\beta \colon \mathcal{T} \to \Sigma$ such that $[\beta, \mathcal{T}] \neq \emptyset$ and the restriction of β to $\mathcal{S}$ is α.

Note that as in the one dimensional setting, the complexity function is translation invariant, meaning that for any $v \in \mathbb{Z}^d$, we have

$$P_X(\mathcal{S}) = P_X(\mathcal{S} + v).$$

2.3. **Expansive subspaces.** An important concept in the study of higher dimensional systems is the notion of an expansive subspace (see Boyle and Lind [3] in particular).

Definition 2.5. Suppose $X \subset \Sigma^{\mathbb{Z}^2}$ is a topological $\mathbb{Z}^2$-system and L is a one-dimensional subspace of $\mathbb{R}^2$. We consider $\mathbb{Z}^2 \subset \mathbb{R}^2$ in the standard way. For $r > 0$, define

$$L_r = \{z \in \mathbb{Z}^2 \colon d(z, L) \leq r\}.$$

We say that the the line L is *expansive* if there exists $r > 0$ such that for any $\eta \in X$, the restriction $\eta|_{L_r}$ extends uniquely to an X-coloring of $\mathbb{Z}^2$. If L is not a horizontal line, we say that L is *right expansive* if there exists $r > 0$ such that for every $\eta \in X$, the restriction $\eta|_{L_r}$ extends uniquely to the half space H^+ consisting of all points of $\mathbb{Z}^2$ to the right of L. *Left expansive* is defined similarly. We call the one-dimensional subspace L *nonexpansive* provided there exist $x, y \in X$ such that $x \neq y$, but $x(i,j) = y(i,j)$ for all (i,j) on one side of the line L. This is equivalent to saying L either fails to be left expansive or fails to be right expansive.

Note that if L is non-horizontal in the u,v-plane, the notions of "right" and "left" are well defined: we set the right side H^+ of L to be the component of $\mathbb{R}^2 \setminus L$ containing the positive horizontal axis $u > 0,\ v = 0$ and define the left side H^- of L to be the other component of $\mathbb{R}^2 \setminus L$. A priori, this definition gives a particular role to the horizontal subspace, but in our setting it turns out that the horizontal is always expansive and so this is not restrictive.

Our definition of nonexpansive is equivalent to the usual definition expansive subspaces, such as that introduced in Boyle and Lind [3].

Note that if L is right expansive and H^- is the left side of L, then for any $\eta \in X$ the restriction $\eta|_{H^-}$ extends uniquely to $\mathbb{Z}^2$. Similarly, if L is left expansive and H^+ is the right side of L, the restriction $\eta|_{H^+}$ extends uniquely to $\mathbb{Z}^2$.

Proposition 2.6. *Suppose $X \subset \Sigma^{\mathbb{Z}^2}$ is a symbolic $\mathbb{Z}^2$-system and let L be a non-horizontal one-dimensional subspace in the u,v-plane. Suppose there is a convex polygon $P \subset \mathbb{R}^2$ such that*

(1) *There is a finite set $F \subset \mathbb{Z}^2$ such that P is the convex hull of F.*
(2) *There is a unique $e \in F$ which is an extreme point of P and which lies on the right side H^+ of L.*
(3) *For any $\eta \in X$, the restriction of η to $F \setminus \{e\}$ extends uniquely to F.*

Then L is right expansive.

The analogous result for left expansive also holds.

Proof. Setting $L_k = L - (0, k) = \{(x, y - k) \colon (x, y) \in L\}$, then the union $\bigcup_{k \geq 0} L_k$ covers the right side H^+ of L. By positioning the set F such that only the extreme point e lies on L and $F \setminus \{e\}$ lies in H^-, the unique extension property determines the element of Σ that lies at e. By moving the set F along the line L, the coloring of L is now determined. Inductively continuing this for each L_k, we obtain a unique coloring of H^+. Thus for any $\eta \in X$, the restriction of η to H^- extends uniquely to $\mathbb{Z}^2$, meaning that L is right expansive. □

Examples 2.7. (1) If (X, σ) is a shift, for $\phi = \sigma^k,\ k \neq 0$, the line $i = kj$ is neither right nor left expansive, but all other lines are expansive.

(2) (Ledrappier's three dot system [15]) With the alphabet $\Sigma = \{0, 1\}$, consider the subset of $\Sigma^{\mathbb{Z}^2}$ defined by

$$x(i, j) + x(i + 1, j) + x(i, j + 1) = 0 \pmod 2$$

for all $i, j \in \mathbb{Z}$. Other than the horizontal axis, the vertical axis, and the reflected diagonal $y = -x$, every one-dimensional subspace is expansive. None of these three subspaces is expansive, but each of them is either right or left expansive.

(3) With the alphabet $\Sigma = \{0, 1\}$, consider the subset of $\Sigma^{\mathbb{Z}^2}$ defined by

$$x(i, j) + x(i + 1, j + 1) + x(i - 1, j + 2) = 0 \pmod 2$$

for all $i, j \in \mathbb{Z}$. It is not difficult to see that the subspaces parallel to the sides of the triangle with vertices $(0, 0), (1, 1)$, and $(-1, 2)$ each fail to be one of left or right expansive (but not both). All other one-dimensional subspaces are expansive.

3. The spacetime of a shift

3.1. **ϕ-coding.** We continue to assume that (X, σ) is an infinite shift over the finite alphabet Σ.

Definition 3.1. If $\phi \in \mathrm{Aut}(X, \sigma)$ is an automorphism, we say that a subset $A \subset \mathbb{Z}$ *ϕ-codes* (or simply *codes* if ϕ is clear from context) a subset $B \subset \mathbb{Z}$ if for any $x, y \in X$ satisfying $x[a] = y[a]$ for all $a \in A$, it follows that $\phi(x)[b] = \phi(x)[b]$ for all $b \in B$.

We remark that if $\phi \in \mathrm{Aut}(X, \sigma)$ is an automorphism then, as ϕ is determined by a block code of some range (say R), the ray $(-\infty, 0]$ ϕ-codes the ray $(-\infty, -R]$. Similarly the ray $[0, \infty)$ ϕ-codes the ray $[R, \infty)$. Of course, it could be the case that $(-\infty, 0]$ ϕ-codes a larger ray than $(-\infty, -R]$. This motivates the following definition:

Definition 3.2. If $\phi \in \mathrm{Aut}(X, \sigma)$ and $n \geq 0$, let $W^+(n, \phi)$ be the smallest element of $\mathbb{Z}$ such that the ray $[W^+(n, \phi), \infty)$ is ϕ^n-coded by $[0, \infty)$ meaning that if x and y agree on $[0, \infty)$, then necessarily $\phi^n(x)$ and $\phi^n(y)$ agree on $[W^+(n, \phi), \infty)$ and this is the largest ray with that property. Similarly $W^-(n, \phi)$ is the largest element of $\mathbb{Z}$ such that the ray $(-\infty, W^-(n, \phi)]$ is ϕ^n-coded by $(-\infty, 0]$. When ϕ is clear from the context, we omit it from the notation and denote $W^+(n,\phi)$ and $W^-(n,\phi)$ by $W^+(n)$ and $W^-(n)$, respectively.

Note that for $n \geq 1$ we have $W^+(n,\phi) = W^+(1,\phi^n)$ and $W^-(n,\phi) = W^-(1,\phi^n)$. These quantities have been studied in [19, 20] in order to define Lyapunov exponents for cellular automata, to study the speed of propagation of perturbations with respect to a shift invariant measure. They use this to give bounds on the entropy of the measure in terms of these (left and right) Lyapunov exponents. We do not consider the role of an invariant measure in this article, but we give an estimate for topological entropy closely related to a result of [20] (see our Theorem 5.13 below).

We check that $W^+(n,\phi)$ and $W^-(n,\phi)$ are well-defined:

Lemma 3.3. *If X is infinite, then $W^+(n,\phi) > -\infty$ and $W^-(n,\phi) < \infty$.*

Proof. It suffices to consider the case $n = 1$, as the general case follows by replacing ϕ with ϕ^n. Suppose $W^+(1,\phi) = -\infty$ and let R be the range of ϕ^{-1}. Since $W^+(1,\phi) = -\infty$, in particular the ray $[0,\infty)$ ϕ-codes the ray $[-R-1,\infty)$. But the $[-R-1,\infty)$ also ϕ^{-1}-codes $[-1,\infty)$. It follows that if $x, y \in X$ and $x[0,\infty) = y[0,\infty)$, then $x[-1,\infty) = y[-1,\infty)$.

We claim that there exists $M \in \mathbb{N}$ such if $x, y \in X$ and $x[0,M] = y[0,M]$, then $x[-1,M] = y[-1,M]$. For contradiction, suppose not. Then there exist sequences (x_n) and (y_n) of points in X such that $x_n[0,n] = y_n[0,n]$, but $x_n[-1,n] \neq y_n[-1,n]$. Passing to a subsequence if necessary, we can assume that $x_{n+1}[-1,n] = x_n[-1,n]$ and $y_{n+1}[-1,n] = y_n[-1,n]$ for all $n \in \mathbb{N}$. Let $z_x, z_y \in X$ be such that $z_x[-1,n] = x_n[-1,n]$ and $z_y[-1,n] = y_n[-1,n]$ for all $n \in \mathbb{N}$. Then $z_x[0,\infty) = z_y[0,\infty)$ but $z_x[-1] \neq z_y[-1]$, a contradiction. This proves the claim and shows that the integer M exists. A similar argument shows that there exists $M' \in \mathbb{N}$ such that for any $x, y \in X$, if $x[-M',0] = y[-M',0]$ then $x[-M',1] = y[-M',1]$. Finally observe that if $x, y \in X$ and if $x[-M',M] = y[-M',M]$ then $x = y$ by inductively applying the rules that $x[n,n+M]$ codes $x[n,n+M+1]$ and $x[n-M',n]$ codes $x[n-M'-1,n]$. Consequently X contains at most $2^{M+M'+1}$ points, a contradiction. Therefore $W^+(1,\phi) > -\infty$.

The argument that $W^-(1,\phi) < \infty$ is similar. □

Corollary 3.4. *If X is infinite, then there exist $x, y \in X$ such that $x[0,\infty) = y[0,\infty)$ but $x[-1] \neq y[-1]$, and there exist $w, z \in X$ such that $w(-\infty,0] = z(-\infty,0]$ but $w[1] \neq z[1]$.*

Proof. Taking ϕ to be the identity in Lemma 3.3, the statement follows. □

By Lemma 3.3, the function $\Theta_n^+ \colon \Sigma^{[0,\infty)} \to \Sigma^{[W^+(n,\phi),\infty)}$ defined by

$$\Theta_n^+(x[0,\infty)) = \phi^n(x)[W^+(n,\phi),\infty)$$

is well defined for all $n \geq 0$, as is the analogous function $\Theta_n^- \colon \Sigma^{(-\infty,0]} \to \Sigma^{(-\infty,W^-(n,\phi)]}$. These functions are continuous:

Proposition 3.5. *The functions Θ_n^+ and Θ_n^- are continuous. In particular, there exists $k = k(n,\phi) > 0$ such that $[0,k]$ ϕ^n-codes $\{W^+(n,\phi)\}$ and $[-k,0]$ ϕ^n-codes $\{W^-(n,\phi)\}$.*

Proof. Assume Θ_n^+ is not continuous. Then there exist x_j and y in X and $r \geq W^+(n,\phi)$ such that $x_j[0,m_j] = y[0,m_j]$, for a sequence $\{m_j\}$ with $\lim_{j\to\infty} m_j = \infty$, and such that $\phi(x_n)[r] \neq \phi(y)[r]$. By passing to a subsequence, w can assume that there exists $z \in X$ with $\lim_{n\to\infty} x_n = z$. Clearly $z[0,\infty) = y[0,\infty)$ and hence $\phi^n(z)[W^+(n,\phi),\infty) = \phi(y)[W^+(n,\phi),\infty)$. In particular, $\phi(z)[r] = \phi(y)[r]$, and so by continuity of ϕ we conclude that $\lim_{n\to\infty} \phi(x_n)[r] = \phi(z)[r] = \phi(y)[r]$. But since $\phi(x_n)[r] \neq \phi(y)[r]$, we also have that $\lim_{n\to\infty} \phi(x_n)[r] \neq \phi(y)[r]$, a contradiction. Thus Θ_n^+ is continuous, and a similar argument shows that Θ_n^- is continuous. □

3.2. **The spacetime of ϕ.**

Definition 3.6. If $\phi \in \mathrm{Aut}(X,\sigma)$ is an automorphism, its *ϕ-spacetime* $\mathcal{U} = \mathcal{U}(\phi)$ is a $\mathbb{Z}^2$-system, together with a preferred basis for $\mathbb{Z}^2$ which we call "horizontal" or the abcissa and "vertical" or the ordinate, that is defined to be the closed subset of $\Sigma^{\mathbb{Z}^2}$ defined by $x \in \mathcal{U}$ if and only if there is $x_0 \in X$ with $\phi^j(x_0)[i] = x(i,j)$ for all $(i,j) \in \mathbb{Z}^2$.

Thus the rows of $\mathcal{U}$ are elements of X with row n equal to ϕ of row $n-1$. There is an action of $\mathbb{Z}^2$ on $\mathcal{U}$ given by having (i,j) shift i times in the horizontal direction and j times in the vertical direction. A vertical shift by j can also be viewed as applying ϕ^j to each row of $\mathcal{U}$.

It follows immediately from the definition of expansiveness (Definition 2.5) that the horizontal axis in a spacetime $\mathcal{U}$ is always an expansive line for the $\mathbb{Z}^2$-system $\mathcal{U}$ with the $\mathbb{Z}^2$-action by translations.

Note that given a spacetime $\mathcal{U}$ (including the preferred basis of $\mathbb{Z}^2$), one can extract the shift (X,σ) by taking X to be the Σ-colorings of $\mathbb{Z}$ obtained by restricting the colorings in $\mathcal{U}$ to the i-axis ($j=0$). Likewise, one can extract the automorphism ϕ by using the fact that if $y \in \mathcal{U}$ and $x \in X$ is given by $x[i] = y[i,0]$, then $\phi(x)[i] = y[i,1]$.

A concept somewhat more general than our notion of spacetime is defined in Milnor [17] and referred to as the *complete history* of a cellular automaton.

We say that spacetimes $\mathcal{U}$ and $\mathcal{U}'$, which share the same alphabet Σ, are *isomorphic* if there is a homeomorphism $h : \mathcal{U} \to \mathcal{U}'$ such that

$$h(z)(i',j') = z(i,j),$$

where the isomorphism of $\mathbb{Z}^2$ for which $(i,j) \mapsto (i',j')$ is given by sending the preferred basis of $\mathbb{Z}^2$ for $\mathcal{U}$ to the preferred basis of $\mathcal{U}'$. We remark that it is straightforward to check that $\phi, \phi' \in \mathrm{Aut}(X)$ are conjugate automorphisms (see Definition 2.1) if and only if their respective spacetimes with the obvious preferred bases are isomorphic.

We extend definition 3.1 of coding to a spacetime:

Definition 3.7. If $\mathcal{U}$ is a topological $\mathbb{Z}^2$-system, we say that a subset $A \subset \mathbb{Z}^2$ *codes* a subset $B \subset \mathbb{Z}^2$ if for any $x, y \in \mathcal{U}$ satisfying $x(i,j) = y(i,j)$ for all $(i,j) \in A$, it follows that $x(i',j') = y(i',j')$ for all $(i',j') \in B$. Equivalently if x and y differ at some point of B, they also differ at some point of A.

Definition 3.8 (**Light Cone**)**.** The *future light cone* $\mathcal{C}_f(\phi)$ *of* ϕ is defined to be

$$\mathcal{C}_f(\phi) = \{(i,j) \in \mathbb{Z}^2 \colon W^-(j,\phi) \le i \le W^+(j,\phi),\ j \ge 0\}$$

The *past light cone* $\mathcal{C}_p(\phi)$ *of* ϕ is defined to be $\mathcal{C}_p(\phi) = -\mathcal{C}_f(\phi)$. The *full light cone* $\mathcal{C}(\phi)$ is defined to be $\mathcal{C}_f(\phi) \cup C_p(\phi)$.

The rationale for this terminology is that if $x \in X$ and $j > 0$, then a change in the value of $x(0)$ (and no other changes) can only cause a change in $\phi^j(x)[i]$ if (i,j) lies in the future light cone of ϕ. Similarly a change in the value of $\phi^{-j}(x)[i]$ can only affect $x[0]$ if (i,j) lies in the past light cone of ϕ.

The light cone is naturally stratified into levels: define the n^{th} *level of* $\mathcal{C}(\phi)$ to be the set

(3.1) {def:I} $$\mathcal{I}(n,\phi) := \{i \in \mathbb{Z} \colon (i,n) \in \mathcal{C}(\phi)\}.$$

In Corollary 3.25 below, we show that if σ is a subshift of finite type and n is large, then the horizontal interval in the light cone at level $-n$, which by definition is $\mathcal{I}(-n,\phi)$, is the unique minimal interval which ϕ^n-codes $\{0\}$, provided ϕ has infinite order in $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$.

In general, it is not clear what the relationship between $\mathcal{C}(\phi)$ and $\mathcal{C}(\phi^{-1})$ is. However there are some restrictions given in Part (5) of Propostion 3.13.

Remark 3.9. A comment about notation is appropriate here. We are interested in subsets of the i,j-plane. Our convention is that i is the abscissa, or first coordinate, and we consider the i-axis to be horizontal. Likewise j is the ordinate, or second coordinate, and we consider the j-axis to be vertical. However some subsets of the plane we consider are naturally described as graphs of a function $i = f(j)$. For example, we frequently consider lines given by an equation like $i = \alpha j,\ j \in \mathbb{R}$, and think of α as a "slope."

The goal of this section is to study the asymptotic behavior of $W^+(j,\phi)$ and $W^-(j,\phi)$ for a fixed $\phi \in \mathrm{Aut}(X)$. We start by recalling Fekete's Lemma, which is then applied to show that the sequence $W^+(n) = W^+(n,\phi)$ for $n \ge 0$ is subadditive.

Lemma 3.10 (Fekete's Lemma [10])**.** *If the sequence* $a_n \in \mathbb{R},\ n \in \mathbb{N}$*, is subadditive (meaning that* $a_n + a_m \ge a_{m+n}$ *for all* $m,n \in \mathbb{N}$*), then*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\ge 1} \frac{a_n}{n}.$$

We note a simple, but useful, consequence of this: if $s(n)$ is subadditive, and if $\lim_{n\to\infty} \dfrac{s(n)}{n} \ge 0$, then $s(n) \ge 0$ for all $n \ge 1$ as otherwise $\inf_{m\ge 1} \dfrac{s(m)}{m}$ would be negative.

Lemma 3.11. *If* $\phi,\psi \in \mathrm{Aut}(X,\sigma)$ *then* $W^+(1,\phi\psi) \le W^+(1,\phi) + W^+(1,\psi)$ *and similarly* $W^-(1,\phi\psi) \ge W^-(1,\phi) + W^-(1,\psi)$*. In particular the sequences* $\{W^+(n,\phi)\}$ *and* $\{-W^-(n,\phi)\},\ n \ge 0$*, are subadditive.*

Proof. The ray $[0,\infty)$ ψ-codes $[W^+(1,\psi),\infty)$ and the ray $[W^+(1,\psi),\infty)$ ϕ-codes $[W^+(1,\phi) + W^+(1,\psi),\infty)$. Hence $[0,\infty)$ $\phi\psi$-codes $[W^+(1,\phi) + W^+(1,\psi),\infty)$ so $W^+(1,\phi\psi) \le W^+(1,\phi) + W^+(1,\psi)$. This proves the first assertion.

Replacing ϕ by ϕ^m and ψ by ϕ^n in this inequality gives

$$W^+(1,\phi^{n+m}) \leq W^+(1,\phi^m) + W^+(1,\phi^n).$$

Since for $n \geq 1$ we have $W^+(n,\phi) = W^+(1,\phi^n)$ we conclude that $W^+(m+n,\phi) \leq W^+(m,\phi)+W^+(n,\phi)$, so $\{W^+(n,\phi)\}$ is subadditive. The proof for W^- is similar. □

We now want to consider two quantities which measure the asymptotic behavior of $W^{\pm}(n,\phi)$. These quantities (and other closely related ones) have been considered in [19, 20] in the context of measure preserving cellular automata and are referred to there as Lyapunov exponents of the automaton. If we fix ϕ and abbreviate $W^+(n,\phi)$ by $W^+(n)$ then Fekete's Lemma and Lemma 3.11, imply the limit $\lim_{n\to\infty} \frac{W^+(n)}{n}$ exists.

Definition 3.12. We define

$$\alpha^+(\phi) := \lim_{n\to\infty} \frac{W^+(n)}{n}$$

and

$$\alpha^-(\phi) := \lim_{n\to\infty} \frac{W^-(n)}{n}.$$

Note that the limit $\alpha^+(\phi)$ is finite, since if $D \geq \max\{\text{range}(\phi), \text{range}(\phi^{-1})\}$, then for $j \geq 0$ we have $|W^+(j))| \leq Dj$ (and $|W^-(j))| \leq Dj$). As a consequence, we conclude that

$$W^+(n) = n\alpha^+(\phi) + \text{o}(n). \tag{3.2}$$

which describes an important asymptotic property of the right light cone boundary function $W^+(n)$ used in the proof of Theorem 4.4 below, which says that if $\alpha^+ = \alpha^+(\phi)$, then the line $i = \alpha^+ j$ is a nonexpansive subspace of $\mathbb{R}^2$ for the spacetime of ϕ.

Similarly, we can consider $W^-(n)$ and obtain a second nonexpansive subspace namely the line $x = \beta y$ where

$$\beta = \alpha^-(\phi) := \lim_{n\to\infty} \frac{W^-(n)}{n}.$$

As a consequence, we conclude that the left light cone boundary function satisfies

$$W^-(n) = n\alpha^-(\phi) + \text{o}(n). \tag{3.3}$$

We list some elementary properties of the limits $\alpha^+(\phi)$ and $\alpha^-(\phi)$:

Proposition 3.13. *If $\phi \in \text{Aut}(X,\sigma)$ then*

(1) *For all $k \in \mathbb{Z}$, $\alpha^-(\sigma^k\phi) = \alpha^-(\phi) + k$ and $\alpha^+(\sigma^k\phi) = \alpha^+(\phi) + k$.*
(2) *For all $m \in \mathbb{N}, \alpha^+(\phi^m) = m\alpha^+(\phi)$ and $\alpha^-(\phi^m) = m\alpha^-(\phi)$*
(3) *If X is infinite, then $\alpha^-(\phi) \leq \alpha^+(\phi)$*
(4) *If $\phi, \psi \in \text{Aut}(X,\sigma)$ are commuting automorphisms then*

$$\alpha^+(\phi\psi) \leq \alpha^+(\phi) + \alpha^+(\psi) \text{ and } \alpha^-(\phi\psi) \geq \alpha^-(\phi) + \alpha^-(\psi).$$

(5) *If X is infinite, then*

$$\alpha^+(\phi) + \alpha^+(\phi^{-1}) \geq 0 \text{ and } \alpha^-(\phi) + \alpha^-(\phi^{-1}) \leq 0.$$

Proof. Since

$$W^+(n,(\sigma^k\phi)) = W^+(1,\sigma^{nk}\phi^n) = W^+(1,\phi^n) + nk = W^+(n,\phi) + nk,$$

property (1) follows. Since

$$\lim_{n\to\infty} \frac{W^+(mn,\phi)}{n} = m \lim_{n\to\infty} \frac{W^+(mn,\phi)}{mn} = m\alpha^+(\phi),$$

property (2) follows.

To show (3), we proceed by contradiction and assume that $\alpha^-(\phi) > \alpha^+(\phi)$. Since $W^+(n) = n\alpha^+(\phi) + \mathrm{o}(n)$ and $W^-(n) = n\alpha^-(\phi) + \mathrm{o}(n)$, the assumption that $\alpha^-(\phi) > \alpha^+(\phi)$ implies that $W^-(n) > W^+(n)$ for all sufficiently large n. By Proposition 3.5, Θ_n^+ is continuous and so for each fixed $n \in \mathbb{N}$, there exists $R > 0$ such that the interval $[0,R]$ ϕ^n-codes $\{W^+(n)\}$. Therefore the interval $[0,R+t]$ ϕ^n-codes the interval $[W^+(n), W^+(n)+t-1]$. Similarly there exists $R' > 0$ such that the interval $[-R',0]$ ϕ^n codes $\{W^-(n)\}$ and so the interval $[0,R'+t]$ ϕ^n-codes the interval $[W^-(n)+R', W^-(n)+R'+t-1]$. Therefore for $t > W^+(n) - W^-(n) + R' - 1$, the interval $[0,R+t]$ ϕ^n-codes the interval $[W^+(n), W^-(n)+R+t]$ (in other words, the two intervals above overlap and so their union is an interval). For any such t, an interval of length $R+t+1$ ϕ^n-codes an interval of length $W^-(n)-W^+(n)+R+t+1 > R+t+1$. However this means that

$$P_X(R+t+1) \geq P_X(R+t+1+W^-(n)-W^+(n)),$$

as every word of length $R+t+1+W^-(n)-W^+(n)$ is ϕ^n-determined by some word of length $R+t+1$. Since $P_X(n)$ is a nondecreasing function, we have $P_X(R+t+1) = P_X(R+t+1+W^-(n)-W^+(n))$. Therefore any word in $\mathcal{L}(X)$ of length $R+t+1$ extends uniquely to the right to a word of greater length, namely of length $R+t+1+W^-(n)-W^+(n)$. Hence it extends uniquely to the infinite ray to the right. Similarly it extends uniquely to the infinite ray to the left. It follows that X contains at most $P_X(R+t+1)$ points, contradicting our standing assumption that X is infinite. This establishes (3).

To prove (4) we note that

$$\begin{aligned} W^+(n,\phi\psi) &= W^+(1,(\phi\psi)^n) = W^+(1,\phi^n\psi^n) \\ &\leq W^+(1,\phi^n) + W^+(1,\psi^n) = W^+(n,\phi) + W^+(n,\psi). \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \frac{W^+(n,\phi\psi)}{n} \leq \lim_{n\to\infty} \frac{W^+(n,\phi)}{n} + \lim_{n\to\infty} \frac{W^+(n,\psi)}{n}$$

giving the inequality of item (4). The result for α^- is similar.

Item (5) follows immediately from (4) if we replace ψ with ϕ^{-1}, since $\alpha^+(id) = \alpha^-(id) = 0$. □

Other than the restriction that $\alpha^-(\phi) \leq \alpha^+(\phi)$, any rational values can be taken on for some automorphism of the full shift:

Example 3.14. We show that given rationals $r_1 \leq r_2$, there is a full shift (X, σ) with an automorphism ϕ such that $\alpha^-(\phi) = r_1$ and $\alpha^+(\phi) = r_2$.

Suppose that $r_2 = p_2/q_2 \geq 0$. Consider X_2 the Cartesian product of q_2 copies of the full two shift $\sigma\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$. Define an automorphism ϕ_0 by having it cyclically permuting the copies of $\{0,1\}^{\mathbb{Z}}$ and perform a shift on one of them. Then $\phi_0^{q_2} = \sigma_2\colon X_2 \to X_2$ is the shift (indeed a full shift on an alphabet of size 2^{q_2}). Since $\alpha^+(\sigma_2) = \alpha^-(\sigma_2) = 1$, it follows from parts (1) and (2) of Proposition 3.13 that $\alpha^+(\phi_0) = \alpha^-(\phi_0) = 1/q_2$. Setting $\phi_2 = \phi_0^{p_2}$, we have that $\alpha^+(\phi_2) = p_2\alpha^+(\phi_2) = p_2/q_2 = r_2$. Similarly $\alpha^-(\phi_2) = r_2$. If $r_2 = -p_2/q_2 < 0$ we can do the same construction, defining ϕ_0 to cyclically permute the copies of Σ_2 but use the inverse shift (instead of the shift) on one of the copies. Then $\phi^{q_2} = \sigma_2^{-1}\colon X_2 \to X_2$. In this way we still construct ϕ_2 with $\alpha^+(\phi_2) = \alpha^-(\phi_2) = r_2$.

By the same argument we can construct an automorphism ϕ_1 of (X_1, σ_1) such that $\alpha^+(\phi_1) = \alpha^-(\phi_1) = r_1$. Taking X to be the Cartesian product $X_1 \times X_2$ and considering the (full) shift $\sigma = \sigma_1 \times \sigma_2\colon X \to X$, and the automorphism $\phi = \phi_1 \times \phi_2$, it is straightforward to check that $\alpha^+(\phi) = \alpha^+(\phi_2) = r_2$ and $\alpha^-(\phi) = \alpha^-(\phi_1) = r_1$.

In light of the work of the work on Lyapunov exponents for cellular auomata, it is natural to ask for a general σ and $\phi \in Aut(X, \sigma)$ a what conditions on ϕ and/or σ suffice for the existence of a σ-invariant ϕ-ergodic measure μ such that $\alpha^{\pm}(\phi)$ are Lyapunov exponents in the sense defined by [19, 20].

3.3. **Two dimensional coding.**

The following lemma essentially appears in [5], and we include the proof for completeness.

Lemma 3.15. *Let $\eta\colon \mathbb{Z}^2 \to \Sigma$ be a coloring and let X_η be the $\mathbb{Z}^2$-subshift obtained by taking the closure of all translations of η. Suppose there exists a nonempty, finite set $\mathcal{S} \subset \mathbb{Z}^2$ such that $P_{X_\eta}(\mathcal{S}) \leq |\mathcal{S}|$. Let $\mathcal{S}' \subseteq \mathcal{S}$ be minimal, with respect to the partial order by inclusion, among all subsets of $\mathcal{S}$ which satisfy $P_{X_\eta}(\mathcal{S}') \leq |\mathcal{S}'|$. Then*

(1) $|\mathcal{S}'| > 1$*;*
(2) *If $x \in \mathcal{S}'$, every legal X_η-coloring of $\mathcal{S}' \backslash \{x\}$ extends uniquely to an X_η-coloring of $\mathcal{S}'$.*

Proof. The first statement follows immediately from the fact that the X_η-complexity of any one-point set is the number of colors in Σ. For the second, suppose $\mathcal{S} \subset \mathbb{Z}^2$ is such that $P_{X_\eta}(\mathcal{S}) \leq |\mathcal{S}|$. Let $\mathcal{S}' \subset \mathcal{S}$ be minimal, with respect to the partial order by inclusion, among all subsets of $\mathcal{S}$ that satisfy $P_{X_\eta}(\mathcal{S}') \leq |\mathcal{S}'|$. Then by part one, we have that $|\mathcal{S}'| > 1$. Fix $x \in \mathcal{S}'$. By minimality, we have $P_{X_\eta}(\mathcal{S}' \setminus \{x\}) > |\mathcal{S}' \setminus \{x\}| = |\mathcal{S}'| - 1$. On the other hand, $P_{X_\eta}(\mathcal{S}' \setminus \{x\}) \leq P_{X_\eta}(\mathcal{S}') \leq |\mathcal{S}'|$ since every legal X_η-coloring of $\mathcal{S}' \setminus \{x\}$ extends in at least one way to a legal X_η-coloring of $\mathcal{S}'$. Therefore $P_{X_\eta}(\mathcal{S}' \setminus \{x\}) = P_{X_\eta}(\mathcal{S}')$, which means that there is no legal X_η-coloring of $\mathcal{S}' \setminus \{x\}$ that extends in more than one way to an η-coloring of $\mathcal{S}'$. □

Remark 3.16. We thank Samuel Petite for suggesting the short proof of the following lemma (in an earlier version of this paper we had a longer proof of this lemma).

Lemma 3.17. *Let $\varphi \in \mathrm{Aut}(X)$ and suppose that there exists K such that* $\mathrm{range}(\varphi^n) \leq K$ *for infinitely many n. Then φ has finite order.*

Proof. There are only finitely many block maps of range $\leq K$ and so, by the pigeonhole principle, there exist $0 < m < n$ such that $\varphi^m = \varphi^n$. It follows that φ^{n-m} is the identity. □

Recall that the interval $\mathcal{I}(n, \phi)$ is defined in equation (3.1) to be $\{i \in \mathbb{Z} \colon (i, n) \in \mathcal{C}(\phi)\}$. Thus for $n \in \mathbb{N}$, we have $|\mathcal{I}(-n, \phi)| = |W^+(n, \phi) - W^-(n, \phi)|$ is the width of the n^{th} level of the light cone for ϕ.

Lemma 3.18. *Suppose ϕ is an automorphism of the shift (X, σ) and $n \geq 0$. If J is any interval in $\mathbb{Z}$ which ϕ^n-codes $\{0\}$, then $J \supset \mathcal{I}(-n, \phi)$.*

Proof. If the interval $J = [a, b]$ ϕ^n-codes $\{0\}$, then $[a, \infty)$ ϕ^n-codes $[0, \infty)$, andso $[0, \infty)$ ϕ^n-codes $[-a, \infty)$. It follows that $-a \geq W^+(n, \phi)$ and hence $a \leq -W^+(n, \phi)$. Similarly $b \geq -W^-(n, \phi)$ and so $\mathcal{I}(-n, \phi) \subset [a, b]$. □

Lemma 3.19. *Assume ϕ is an automorphism with infinite order of a shift of finite type (X, σ) and suppose that*

$$\lim_{n \to \infty} \left|\mathcal{I}(-n, \phi)\right| = +\infty.$$

Then there is n_0 such that whenever $n \geq n_0$, the interval $\mathcal{I}(-n, \phi)$ ϕ^n-codes $\{0\}$. Moreover, if σ is a full shift we can take n_0 to be 0 and the hypothesis $\lim_{n\to\infty} |\mathcal{I}(-n, \phi)| = \infty$ is unnecessary.

Proof. Suppose that ϕ has infinite order and that σ is a subshift of finite type. Then by Proposition 2.3, there exists $m_0 \geq 0$ such that if w is a word of length at least m_0 and if $w_1^- w w_1^+$ and $w_2^- w w_2^+$ are elements of X for some semi-infinite words $w_i^\pm$, then both $w_1^- w w_2^+$ and $w_2^- w w_1^+$ are elements of X. Clearly $m_0 = 0$ suffices if σ is a full shift.

By hypothesis,

$$\lim_{n \to \infty} \left|\mathcal{I}(-n, \phi)\right| = \lim_{n \to \infty} \left|W^+(n, \phi) - W^-(n, \phi)\right| = +\infty,$$

and so we can choose n_0 such that the length of $\mathcal{I}(-n, \phi)$ is greater than m_0 when $n \geq n_0$. Suppose $n \geq n_0$ and that $x, y \in X$ agree on the interval $\mathcal{I}(-n, \phi)$. We show that $\phi^n(x)[0] = \phi^n(y)[0]$. Let $w = x[-W^+(n, \phi), -W^-(n, \phi)] = y[-W^+(n, \phi), -W^-(n, \phi)]$ and define $w_i^\pm$ by $x(-\infty, \infty) = w_1^- w w_1^+$ and $y(-\infty, \infty) = w_2^- w w_2^+$. Then $w_1^- w w_2^+$ is an element of X satisfying $x[i] = w_1^- w w_2^+[i]$ for all $i \geq -\omega^+(\phi^n)$ and $y[i] = w_1^- w w_2^+[i]$ for all $i \leq -\omega^+(\phi^n)$. It follows that $\phi^n(x)[0] = \phi^n(w_1^- w w_2^+)[0] = \phi^n(y)[0]$. Hence $\{0\}$ is ϕ^n-coded by $[-W^+(n, \phi), -W^-(n, \phi)]$.

□

Definition 3.20. Let X be a subshift and let $\phi \in \mathrm{Aut}(X, \sigma)$. Define $r(n, \phi)$ to be the minimal width of an interval which ϕ^n-codes $\{0\}$.

Lemma 3.21. *Suppose (X, σ) is a subshift of finite type. Then there is a constant $C(\phi)$ such that $|\mathcal{I}(-n, \phi)| \leq r(n, \phi) \leq |\mathcal{I}(-n, \phi)| + C(\phi)$. If X is a full shift we can take $C(\phi) = 0$.*

Proof. The first inequality follows immediately from Lemma 3.18. We prove the second inequality by contradiction. Thus suppose that for any C, there exist infinitely many $n \in \mathbb{N}$ and points $x_{C,n} \neq y_{C,n}$ which agree on the interval $[-W^+(n,\phi), -W^-(n,\phi)+C]$ but are such that $\phi^n(x_{C,n})[0] \neq \phi^n(y_{C,n})[0]$.

Recall from Proposition 2.3 that there exists a constant n_0 (depending on the subshift X) such that if $x, y \in X$ agree for n_0 consecutive places, say $x[i] = y[i]$ for all $p \leq i < p+n_0$, then the $\mathbb{Z}$-coloring whose restriction to $(-\infty, p+n_0-1]$ coincides with that of x and whose restriction to $[p+n_0, \infty)$ coincides with that of y, is an element of X.

Choose $C > n_0$. By assumption, there exist infinitely many $n \in \mathbb{N}$ and points $x_n, y_n \in X$ which agree on $[-W^+(n,\phi), -W^-(n,\phi)+C]$ but are such that $\phi^n(x_n)[0] \neq \phi^n(y_n)[0]$. Let $z \in X$ be the $\mathbb{Z}$-coloring whose restriction to $(-\infty, -W^-(n,\phi)+C]$ coincides with x_n and whose restriction to $[-W^-(n,\phi)+C+1, \infty)$ coincides with y_n. Then since $C > n_0$, we have that $z \in X$. Since z agrees with y_n on $[-W^+(n,\phi), \infty)$, it follows that $(\phi^n z)[0] = (\phi^n y_n)[0]$. On the other hand, $(\phi^n z)[0] = (\phi^n x_n)[0]$, since z agrees with x_n on $(-\infty, -W^-(n,\phi)]$. But this contradicts the fact that $(\phi^n x_n)[0] \neq (\phi^n y_n)[0]$, and so $C(\phi)$ exists. $\square$

Lemma 3.22. *Suppose X is a subshift of finite type and $\phi \in \mathrm{Aut}(X,\sigma)$. If*

$$\liminf_{n\to\infty} |\mathcal{I}(-n,\phi)| < \infty,$$

then any nonexpansive line in the spacetime of ϕ has rational slope.

Proof. By hypothesis, there exists M such that $|\mathcal{I}(-n,\phi)| < M$ for infinitely many n. By Lemma 3.21, there is a constant $C(\phi)$ such that $r(n,\phi) < M + C(\phi)$ for infinitely many n. Let $\{n_1, n_2, \dots, n_k\}$, where $k = P_X(M+C(\phi)) + 2M + 2C(\phi) + 1$, be such that the interval $\mathcal{I} := [-M-C(\phi), M+C(\phi)]$ ϕ^{n_i}-codes $\{m_i\}$ for some $m_i \in \mathbb{N}$. Define

$$\mathcal{T} := \{(x,0)\colon -M-C(\phi) \leq x \leq M+C(\phi)\} \cup \{(m_i,n_i)\} \subset \mathbb{Z}^2$$

and let $\mathcal{U}$ be the spacetime of ϕ. Then the set $\{(x,0)\colon -M-C(\phi) \leq x \leq M+C(\phi)\}$ $\mathcal{U}$-codes $\mathcal{T}$. In particular, the $P_{\mathcal{U}}(\mathcal{T}) \leq |\mathcal{T}|$. By Lemma 3.15, if $\mathcal{T}' \subset \mathcal{T}$ is minimal (with respect to the partial order by inclusion) among all subsets of $\mathcal{T}$ which satisfy $P_{\mathcal{U}}(\mathcal{T}') \leq |\mathcal{T}'|$, then for any $x \in \mathcal{T}'$, the restriction of any legal $\mathcal{U}$-coloring of $\mathcal{T}' \setminus \{x\}$ uniquely extends a legal $\mathcal{U}$-coloring of $\mathcal{T}'$.

Let $\mathcal{S} \subset \mathbb{Z}^2$ be the intersection of $\mathbb{Z}^2$ with the convex hull of $\mathcal{T}'$. Let $\partial\mathcal{S}$ denote the boundary of the convex hull of $\mathcal{T}'$ (this is either a line segment in $\mathbb{R}^2$ or a convex polygon). We claim that if L is a line through the origin in $\mathbb{R}^2$ which is not parallel to an edge of $\partial\mathcal{S}$, then L is expansive. Indeed, if L is such a line then there exist two vectors $u, v \in \mathbb{Z}^2$ such that L separates a vertex of $\mathcal{S}+u$ from the remainder of the set, and $\mathcal{S}+v$ separates a vertex of $\mathcal{S}+v$ from the remainder of the set (where the remainder of the set is on a "different side" of L in each case). By construction, any $\mathcal{U}$-legal pattern on $\mathcal{S}$ with one of its vertices removed, extends uniquely to a $\mathcal{U}$-legal coloring of $\mathcal{S}$. In particular, if $B \subset \mathbb{Z}^2$ is a neighborhood of L taken to be sufficiently wide as to contain $\mathcal{S}$, then any $\mathcal{U}$-legal coloring of B extends uniquely to a $\mathcal{U}$-legal coloring of $\mathbb{Z}^2$. It follows from Lemma 2.6 that L is an expansive line. $\square$

Proposition 3.23. *Suppose X is a subshift of finite type and $\phi \in \mathrm{Aut}(X, \sigma)$. If*

$$\liminf_{n\to\infty} |\mathcal{I}(-n, \phi)| < \infty,$$

then ϕ has finite order in $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$.

Proof. Let $\mathcal{U}$ be the spacetime of ϕ. If $\mathcal{U}$ has no nonexpansive line, by a result in Boyle and Lind [3, Theorem 3.7], $\mathcal{U}$ is comprised entirely of doubly periodic $\mathbb{Z}^2$-colorings with uniformly bounded period. Thus ϕ has finite order in $\mathrm{Aut}(X, \sigma)$. Therefore we can assume that $\mathcal{U}$ has at least one nonexpansive line.

Let M and $C(\phi)$ be as in Lemma 3.22 and suppose that L is a nonexpansive line for $\mathcal{U}$. By Lemma 3.22, L has rational slope. Choose $u, v \in \mathbb{Z}^2$ such that $L+u$ passes through one of the endpoints of the interval

$$\mathcal{I} := \{(x, 0)\colon -M - C(\phi) \le x \le M + C(\phi)\}$$

and $L+v$ passes through the other. Let $N \in \mathbb{N}$ be the number of distinct lines parallel to L that intersect an integer point in the strip bounded between $L+u$ and $L+v$. Choose points $\{n_1, \dots, n_k\}$, where $k = N \cdot P_X(M + C(\phi)) + 2NM + 2NC(\phi) + N$, such that any $\mathcal{U}$-legal coloring of $\mathcal{I}$ ϕ^{n_i} codes $\{m_i\}$ for some $m_i \in \mathbb{N}$. Define

$$\mathcal{T} := \mathcal{I} \cup \{(m_i, n_i)\colon 1 \le i \le N \cdot P_X(M + C(\phi)) + 2NM + 2NC(\phi) + N\}.$$

Since any $\mathcal{U}$-legal coloring of $\mathcal{I}$ extends uniquely to a $\mathcal{U}$-legal coloring of $\mathcal{T}$, it follows that $\mathcal{T}$ is contained entirely in the strip bounded by $L+u$ and $L+v$ (otherwise by Proposition 2.6, L would be an expansive line).

Consequently, there is at least one line L' parallel to L which contains at least $P_X(M + C(\phi)) + 2M + 2C(\phi) + 1$ points of $\mathcal{T}$. As there are at most $P(M + C(\phi))$ many $\mathcal{U}$-legal colorings of $\mathcal{T}$ (each is determined by a coloring of $\mathcal{I}$) it follows that $P_{\mathcal{U}}(\mathcal{T}) \le |\mathcal{T}|$. As in Lemma 3.22, let $\mathcal{T}' \subseteq \mathcal{T}$ be minimal (with respect to the partial order by inclusion) among all subsets of $\mathcal{T}$ that satisfy $P_{\mathcal{U}}(\mathcal{T}') \le |\mathcal{T}'|$. Then, as in Lemma 3.17, any element of $\mathcal{U}$ is periodic with period vector parallel to L and uniformly bounded periods. Therefore there exists $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that $\phi^n = \sigma^m$, and the statement of the proposition follows. $\square$

Theorem 3.24. *Assume that ϕ is an automorphism of a shift of finite type (X, σ) and that ϕ has infinite order in $\mathrm{Aut}(X)/\langle\sigma\rangle$. Then there exists n_0 such that whenever $n \ge n_0$, the interval $\mathcal{I}(-n, \phi)$ ϕ^n-codes $\{0\}$. If σ is a full shift, we can take n_0 to be 0.*

Proof. If σ is a full shift, the result follows from Lemma 3.19 and moreover, ϕ has infinite order in $\mathrm{Aut}(X)$. Otherwise, since ϕ has infinite order in $\mathrm{Aut}(X)/\langle\sigma\rangle$, Proposition 3.23 tells us that

$$\lim_{n\to\infty} |\mathcal{I}(-n, \phi)| = +\infty.$$

Thus we can apply Lemma 3.19 to conclude that $\mathcal{I}(-n, \phi)$ ϕ^n-codes $\{0\}$. $\square$

Corollary 3.25. *If ϕ has infinite order in $\mathrm{Aut}(X)/\langle\sigma\rangle$, then for n sufficiently large, $\mathcal{I}(-n,\phi)$ is the unique minimal interval which ϕ^n-codes $\{0\}$.*

Proof. The fact that $\mathcal{I}(-n,\phi)$ ϕ^n-codes $\{0\}$ for large n follows from Theorem 3.24. Minimality and uniqueness follow from Lemma 3.18. □

Question 3.26. *Is the hypothesis that (X,σ) is an SFT necessary in Theorem 3.24?*

4. The light cone and nonexpansive subspaces

The main result of this section is Theorem 4.4: it states that the line $u = \alpha^+(\phi)v$ in the u, v-plane is a nonexpansive subspace of $\mathbb{R}^2$ for the spacetime of ϕ. The analogous statement holds in the other direction: the the line $u = \alpha^-(\phi)v$ in the u, v-plane is a nonexpansive subspace.

4.1. **The deviation function.**

We begin by investigating the properties of the function which measures the deviation of $W^+(n,\phi)$ from $\alpha^+(\phi)n$.

Definition 4.1. For a ϕ-spacetime of a shift (X,σ), define the positive and negative *deviation functions* $\delta^+(n) = \delta^+(n,\phi)$ and $\delta^-(n) = \delta^-(n,\phi)$ by $\delta^+(n) = W^+(n) - n\alpha^+(\phi)$ and $\delta^-(n) = W^-(n) - n\alpha^-(\phi)$

Lemma 4.2. *Suppose $\delta^+(n)$ and $\delta^-(n)$ are the deviation functions associated to ϕ. Then*

1. *The functions $\delta^+(n)$ and $\delta^+(n)$ are subadditive.*
2. *The deviation functions satisfy $\lim_{n\to\infty} \frac{\delta^+(n)}{n} = 0$ and $\lim_{n\to\infty} \frac{\delta^-(n)}{n} = 0$.*
3. *For all $n \geq 0$, we have $\delta^+(n) \geq 0$ and $\delta^-(n) \leq 0$.*

Proof. Since $\delta^+(n)$ is the sum of the subadditive function $W^+(n) = W^+(\phi^n)$ and the additive function $-n\alpha$, part (1) follows. Since

$$\lim_{n\to\infty} \frac{\delta^+(n)}{n} = \lim_{n\to\infty} \frac{W^+(n) - n\alpha^+(\phi)}{n} = \lim_{n\to\infty} \frac{W^+(n)}{n} - \alpha^+(\phi) = 0,$$

part (2) follows.

To see part (3), observe that parts (1) and (2) together with Fekete's Lemma (Lemma 3.10) imply

$$\inf_{n\geq 1} \frac{\delta^+(n)}{n} = 0$$

and so $\delta^+(n) < 0$ is impossible. The analogous results for $\delta^-(n)$ are proved similarly. □

Lemma 4.3. *Let $\mathcal{U}$ be the ϕ-spacetime of (X,σ) and let $\alpha = \alpha^+(\phi)$ and $\delta(n) = \delta^+(n,\phi)$. Suppose that $\alpha \geq 0$ and the deviation $\delta(n)$ is unbounded for $n \geq 0$. Then there exist two sequences $\{x_m\}_{m\geq 1}$ and $\{y_m\}_{m\geq 1}$ in $\mathcal{U}$ such that*

1. *$x_m(i,j) = y_m(i,j)$ for all (i,j) with $-m \leq j \leq 0$ and $i \geq \alpha j$*
2. *$x_m(i,j) = y_m(i,j)$ for all (i,j) with $j \geq 0$ and $i \geq (\alpha + \frac{1}{m})j$*
3. *$x_m(-1,0) \neq y_m(-1,0)$ for all $m \in \mathbb{N}$.*

The analogous result for $\alpha^-(\phi)$ and $\delta^-(n,\phi)$ also holds.

Proof. For notational simplicity, denote $W^+(n)$ by $W(n)$, so $\delta(n) = W(n) - n\alpha$.

We define a piecewise linear $F(t)$ from the set $\{t \in \mathbb{Z} \colon t \geq -m\}$ to $\mathbb{Z}$ and show $W(t) \leq F(t)$ for all $-m \leq t$. We then use this to define x_m, y_m satisfying the three properties.

Given $m \in \mathbb{N}$ and using the facts that $\lim_{k\to\infty} \frac{W(k)}{k} = \alpha$ and $\lim_{k\to\infty} \frac{\delta(k)}{k} = 0$, we can choose $n_0 = n_0(m) > m$ such that

$$\frac{\delta(k)}{k} < \frac{1}{m}$$

for all $k > n_0$. For the moment as m is fixed we suppress the dependence of n_0 on m. By hypothesis, $\delta(k)$ is unbounded above and so we can also choose n_0 so that

$$\delta(n_0) > \delta(j) \text{ for all } 0 \leq j < n_0. \tag{4.1}$$
{dn_0 > dj}

Define a line $i = L(j)$ in the i, j-plane by

$$L(j) = \frac{1}{m}(j - n_0) + \delta(n_0).$$

We claim that the set of j with $\delta(j) \geq L(j)$ is finite. By Lemma 4.2,

$$\lim_{j\to\infty} \frac{\delta(j)}{j - n_0} = \lim_{j\to\infty} \frac{\delta(j)}{j} = 0$$

and so for sufficiently large j,

$$\delta(j) \leq \frac{1}{m}(j - n_0) < \frac{1}{m}(j - n_0) + \delta(n_0) = L(j),$$

since $\delta(n_0) \geq 0$ (by Lemma 4.2). This proves the claim.

Let J be the finite set $\{j \;:\; \delta(j) \geq L(j),\; j \geq 0\}$ and let $S = \{(\delta(j), j) \;:\; j \in J\}$. Note that $S \neq \emptyset$ since $(\delta(n_0), n_0) \in S$.

Let $t_0 = t_0(m) \in \mathbb{N}$ be the value of j with $j \geq n_0$ for which $\delta(j) - L(j)$ is maximal. Then $(\delta(t_0), t_0) \in S$. Since, for the moment m is fixed, we suppress the m and simply write t_0 for $t_0(m)$.

Suppose now that $j \in [n_0, t_0]$ Then since $\delta(t_0) - L(t_0) \geq \delta(j) - L(j) = 0$, it follows that $\delta(t_0) \geq \delta(j) + L(t_0) - L(j) \geq \delta(j)$ since $j \in [n_0, t_0]$ and L is monotonic increasing. Thus we have

$$\delta(t_0) \geq \delta(j) \text{ for all } j \in [n_0, t_0]. \tag{4.2}$$
{eq:dt ge dj}

Let $\alpha_m = \alpha + \frac{1}{m}$ and consider the two lines

$$i = \mathcal{K}(j), \text{ where } \mathcal{K}(j) = \alpha(j - t_0) + W(t_0)$$

and

$$i = \mathcal{L}(j), \text{ where } \mathcal{L}(j) = \alpha_m(j - t_0) + W(t_0).$$

Both lines pass through $(W(t_0), t_0)$.

Define

$$F(j) = \begin{cases} \mathcal{K}(j), & \text{if } 0 \le j \le t_0 \\ \mathcal{L}(j), & \text{if } j \ge t_0. \end{cases} \tag{4.3}$$

(4.3) {def Fj}

We claim that for all $j \ge 0$

$$W(j) \le F(j).$$

We prove this claim by considering two separate ranges of values for j, first $j \ge t_0$, then $0 \le j \le t_0$.

In the range $j \ge t_0$, by the choice of t_0 we have that $\delta(j) - L(j) \le \delta(t_0) - L(t_0)$ if $j \in J$. But the same inequality holds for $j \notin J$ since then $\delta(j) - L(j) < 0$ and $\delta(t_0) - L(t_0) \ge 0$. Thus $\delta(j) \le L(j) + \delta(t_0) - L(t_0)$ for all $j \ge t_0$. Therefore

$$\begin{aligned} W(j) &= \delta(j) + \alpha j \\ &\le L(j) + \delta(t_0) - L(t_0) + \alpha j \\ &= \frac{1}{m}(j - n_0) - \frac{1}{m}(t_0 - n_0) + \delta(t_0) + \alpha j \\ &= \frac{1}{m}(j - t_0) + \delta(t_0) + \alpha t_0 + \alpha(j - t_0) \\ &= \alpha_m(j - t_0) + \delta(t_0) + \alpha t_0 \\ &= \alpha_m(j - t_0) + W(t_0) \\ &= \mathcal{L}(j). \end{aligned}$$

This proves the claim for the first range, i.e.,

$$W(j) \le \mathcal{L}(j) \text{ for } j \ge t_0. \tag{4.4}$$

Next we consider the range $0 \le j \le t_0$. Note if $j \le n_0$ then $W(j) = \delta(j) + \alpha j \le \delta(n_0) + \alpha j$ by Equation (4.1), so $W(j) \le \delta(t_0) + \alpha j$ since $\delta(t_0) \ge \delta(n_0)$ by Equation (4.2). But if $j \in [n_0, t_0]$ then $W(j) = \delta(j) + \alpha j \le \delta(t_0) + \alpha j$ by Equation (4.2). So we conclude $W(j) \le \delta(t_0) + \alpha j$ for all $0 \le j \le t_0$.

Hence in this range

$$\begin{aligned} W(j) &\le \delta(t_0) + \alpha j \\ &= \delta(t_0) + \alpha t_0 + \alpha(j - t_0) \\ &= W(t_0) + \alpha(j - t_0) = \mathcal{K}(j). \end{aligned}$$

Thus we have

$$W(j) \le \mathcal{K}(j) \text{ for } 0 \le j \le t_0 \tag{4.5}$$

Hence Equations (4.4), and (4.5) establish the claim, demonstrating that

$$F(j) \ge W(j) \text{ for all } j \ge 0, \tag{4.6}$$

where

$$F(j) = \begin{cases} \mathcal{K}(j) & \text{if } 0 \le j \le t_0 \\ \mathcal{L}(j) & \text{if } j \ge t_0. \end{cases}$$

We now use this to define the elements x_m and y_m. ¿From the definition of $W^+(n,\phi)$ (which we are denoting $W(n)$), we know that whenever $j \geq 0$ and $u, v \in X$ have the rays $u[0,\infty)$ and $v[0,\infty)$ equal, it follows that the rays $\phi^j(u)[W(j),\infty) = \phi^j(v)[W(j),\infty)$. Equivalently if x and y are the elements in ϕ-spacetime which agree on the ray $\{(i,0) \in \mathbb{Z}^2 \colon i \geq 0\}$, then

(4.7) {i > Wj} $$j \geq 0 \text{ and } i \geq W(j) \text{ implies } x(i,j) = y(i,j).$$

Moreover for each $j \geq 0$, there exist $u_j, v_j \in X$ such that $u_j[0,\infty) = v_j[0,\infty)$ but

$$\phi^j(u_j)(W(j)-1) \neq \phi^j(v_j)(W(j)-1).$$

In particular this means that for $m \in \mathbb{N}$ there exist elements $\hat{x}_m, \hat{y}_m \in \mathcal{U}$ which are equal on the ray $\{(i,0) \in \mathbb{Z}^2 \colon i \geq 0\}$, but such that

(4.8) {W minus 1} $$\hat{x}_m(W(t_0(m))-1, t_0(m)) \neq \hat{y}_m(W(t_0(m))-1, t_0(m)).$$

(Note that the dependence of $t_0 = t_0(m)$ on m is now salient so we return to the more cumbersome notation.) We use translates of $\hat{x}_m$ and $\hat{y}_m$ by the vectors $(W(t_0(m)), t_0(m)) = (\delta(t_0(m)) + \alpha t_0(m), t_0(m))$ to define $x_m, y_m \in \mathcal{U}$. More precisely, define

$$x_m(i,j) = \hat{x}_m(i + W(t_0(m)), j + t_0(m))$$

and

$$y_m(i,j) = \hat{y}_m(i + W(t_0(m)), j + t_0(m)).$$

Note that x_m and y_m agree on the ray $\{(i,0) \in \mathbb{Z}^2 \colon i \geq 0\}$.

We proceed to check properties (1), (2), and (3) of the lemma's conclusion.

From Equation (4.8) and the definition of x_m and y_m we have

$$x_m(-1,0) = \hat{x}_m(W(t_0(m))-1, t_0(m)) \neq \hat{y}_m(W(t_0(m))-1, t_0(m)) = y_m(-1,0),$$

and so (3) follows.

To check (1), suppose $-m \leq j \leq 0$ and $i \geq \alpha j$. Let $i' = i + W(t_0)$ and $j' = j + t_0$ and so $x_m(i,j) = \hat{x}_m(i',j')$ and $y_m(i,j) = \hat{y}_m(i',j')$. Hence if we show that $\hat{x}_m(i',j') = \hat{y}_m(i',j')$, then we have that $x_m(i,j) = y_m(i,j)$, which is the statement of (1). This in turn follows from Equation (4.7) if we show $j' \geq 0$ and $i' \geq W(j')$. We proceed to do so.

Note that since $-m \leq j \leq 0$ and since, by construction, $n_0(m)$ and $t_0(m)$ satisfy $m < n_0(m) < t_0(m))$, we have

$$0 \leq t_0(m) - m \leq t_0(m) + j = j'$$

To show $i' \geq W(j')$ observe that since $i \geq j\alpha$, it follows that $i' = i + W(t_0) \geq j\alpha + W(t_0) = (j' - t_0)\alpha + W(t_0) = \mathcal{K}(j')$. Since $j' = j + t_0(m) \leq t_0(m)$ the definition of F (equation 4.3) shows $\mathcal{K}(j') = F(j')$, and we may apply Equation (4.6) to conclude $i' \geq W(j')$. Then by Equation 4.7 applied to $\hat{x}_m$ and $\hat{y}_m$ at (i',j') we have $\hat{x}_m(i',j') = \hat{y}_m(i',j')$, so $x_m(i,j) = y_m(i,j)$. This completes the proof of property (1).

To check (2), we assume that $j \geq 0$ and $i \geq \alpha_m j$. Again we let $i' = i + W(t_0(m))$ and $j' = j + t_0(m)$ and so $j' \geq t_0(m)$. To show $x_m(i,j) = y_m(i,j)$, it suffices to show

$\hat{x}_m(i',j') = \hat{y}_m(i',j')$ when

$$j' \geq t_0(m) \text{ and } i' \geq \alpha_m j + W(t_0(m))$$

But $\alpha_m j + W(t_0(m)) = \alpha_m(j' - t_0) + W(t_0) = \mathcal{L}(j')$, so we have $j' \geq t_0(m)$ and $i' \geq \mathcal{L}(j')$.

Since $j' \geq t_0(m)$ we conclude from the definition of F (Equation (4.3)) that $F(j') = \mathcal{L}(j')$. So $i' \geq F(j')$ and hence by Equation (4.6), $i' \geq F(j') \geq W(j')$. Since $i' \geq W(j')$ we have $x_m(i',j') = y_m(i',j')$ by Equation 4.7, completing the proof of (2).

The proof of the analogous result for $\alpha^-(\phi)$ and $\delta^-(n,\phi)$ is done similarly. □

4.2. **Nonexpansiveness of light cone edges.**

Theorem 4.4. *Suppose $\phi \in \mathrm{Aut}(X,\sigma)$ and $\alpha^+ = \alpha^+(\phi)$. Then for the spacetime of ϕ, the line $u = \alpha^+ v$ is not left expansive. Similarly if $\alpha^- = \alpha^-(\phi)$, then the the line $u = \alpha^- v$ in $\mathbb{R}^2$ is not right expansive.*

Proof. Let $\mathcal{U}$ be the ϕ-spacetime of (X,σ). Replacing ϕ with $\sigma^k\phi^m$ and using part (1) of Proposition 3.13, without loss of generality we can assume that $\alpha^+(\phi) \geq 0$.

Case 1: bounded deviation. As a first case we assume that the non-negative deviation function δ is bounded. Say $\delta(j) < D$ for some $D > 0$ and all $j \in \mathbb{N}$. Since $\delta(j) \geq 0$ and $\alpha^+ \geq 0$, we have $0 \leq W^+(j,\phi) - \alpha^+ j = \delta(j) < D$.

If we have two elements $x, y \in \mathcal{U}$ satisfying $x(k,0) = y(k,0)$ for $k \geq 0$ then whenever $j \geq 0$ and $i \geq D + \alpha^+ j$, we have $i > W^+(j)$. Hence

$$x(i,j) = y(i,j) \text{ for all } j \geq 0 \text{ and } i \geq D + \alpha^+ j \tag{4.9}$$

(see Equation (4.7)). Thus x and y agree in the part of the upper half space to the right of the line $i = D + \alpha^+ j$.

By the defintion of $W^+(n) = W^+(n,\phi)$ for $n \in \mathbb{N}$ we may choose $\hat{x}_n, \hat{y}_n \in \mathcal{U}$ which agree on the ray $\{(i,0) \in \mathbb{Z}^2 \colon i \geq 0\}$ such that $\hat{x}_n(W^+(n)-1,n) \neq \hat{y}_n(W^+(n)-1,n)$.

We want to create new colorings by translating $\hat{x}_n$ and $\hat{y}_n$ by the vector $(W^+(n),n)$. More precisely for $n \geq 0$ we define $\hat{x}_n$ and $\hat{y}_n$ by $x_n(i,j) = \hat{x}_n(i+W^+(n), j+n)$. Note that $x_n(-1,0) \neq y_n(-1,0)$, since $x_n(-1,0) = \hat{x}_n(W^+(n)-1,n) \neq \hat{y}_n(W^+(n)-1,n) = y_n(-1,0)$.

For all $j \geq -n$ and $i \geq D + \alpha^+ j$, we claim that

$$x_n(i,j) = y_n(i,j).$$

To see this define $i' = i + W^+(n)$ and $j' = j + n$ and so $x_n(i,j) = \hat{x}_n(i',j')$ and $y_n(i,j) = \hat{y}_n(i',j')$. Then

$$\begin{aligned} i' &= i + W^+(n) \\ &\geq D + \alpha^+ j + W^+(n) \\ &= D + \alpha^+ j' + (W^+(n) - \alpha^+ n) \\ &= D + \alpha^+ j' + \delta(n) \\ &\geq D + \alpha^+ j'. \end{aligned}$$

Hence $\hat{x}_n(i',j')$ and $\hat{y}_n(i',j')$ are equal by Equation 4.9 whenever $i \geq D+\alpha^+ j$ and $j \geq -n$ (since $j' \geq 0$ when $j \geq -n$). But $x_n(i,j) = \hat{x}_n(i',j')$ and $y_n(i,j) = \hat{y}_n(i',j')$ so $x_n(i,j) = y_n(i,j)$. Thus x_n and y_n agree at (i,j) if $i \geq D+\alpha^+ j$ and $j \geq -n$.

Since $\mathcal{U}$ is compact we can choose convergent subsequences (also denoted x_n and y_n). Say $\lim x_n = \hat{x}$ and $\lim y_n = \hat{y}$. Then clearly $\hat{x}(-1,0) \neq \hat{y}(-1,0)$ and $\hat{x}(i,j) = \hat{y}(i,j)$ for all $i > D+\alpha^+ j$. So $\hat{x}$ and $\hat{y}$ agree on the half space $H^+ = \{(i,j) \colon i > D+\alpha^+ j\}$. This implies the line $u = \alpha^+ v$ is not left expansive. The case of the line $u = \alpha^- v$ is handled similarly.

Case 2: unbounded deviation. We consider the elements x_m, y_m guaranteed by Lemma 4.3, and recall that they satisfy properties (1)-(3) of the lemma.

Since $\mathcal{U}$ is compact, by passing to subsequences, we can assume that both sequences converge in $\mathcal{U}$, say to $\hat{x}$ and $\hat{y}$. Clearly $\hat{x}(-1,0) \neq \hat{y}(-1,0)$. We claim the colorings $\hat{x}$ and $\hat{y}$ agree on the half space $H^+ = \{(i,j) \colon i > \alpha j\}$ of $\mathbb{Z}^2$. It then follows that the line $u = \alpha v$ is not left expansive (see Definition 2.5).

To prove the claim, note that if $(i,j) \in H^+$, $-m_0 \leq j \leq 0$ and $m \geq m_0$ then $x_m(i,j) = y_m(i,j)$. Hence the limits satisfy $\hat{x}(i,j) = \hat{y}(i,j)$ whenever $(i,j) \in H^+$ and $j \leq 0$. But also if $j > 0$ and $i > \alpha j$, then for some $n_0 > 0$ we have $i \geq (\alpha + \frac{1}{n_0})j$ and it follows that $x_m(i,j) = y_m(i,j)$ whenever $m > n_0$. Hence the limits satisfy $\hat{x}(i,j) = \hat{y}(i,j)$.

The case of the line $u = \alpha^- v$ is handled similarly. □

4.3. **Expansive subspaces.** We want to investigate which one-dimensional subspaces in a spacetime are expansive. Since the horizontal axis in a spacetime is always expansive, we can restrict our attention to lines in $\mathbb{R}^2 = \{(u,v)\}$ given by $u = mv$ where $m \in \mathbb{R}$. (We write the abscissa as a function of the ordinate for convenient comparison with the edges of $\mathcal{A}(\phi)$ which are $u = \alpha^+ v$ and $u = \alpha^- v$.)

Proposition 4.5. *Suppose L is a line in $\mathbb{R}^2$ given by $u = mv$. Then:*

(1) *If $m > \alpha^+(\phi)$, then L is left expansive.*
(2) *If $m < \alpha^-(\phi)$, then L is right expansive.*

Moreover if $m > \max\{\alpha^+(\phi), -\alpha^-(\phi^{-1})\}$ or $m < \min\{\alpha^-(\phi), -\alpha^+(\phi^{-1})\}$, then L is expansive.

Proof. We first consider (1). We show that if $\mathcal{U}$ is the spacetime of ϕ and $x, y \in \mathcal{U}$ agree on the right side of $u = mv$, then they also agree on the left side. Since $m > \alpha^+(\phi)$, the vector $\langle \alpha^+(\phi), 1\rangle$ is not parallel to L and points in the direction from the right side of L to the left side.

Let $W^+(n) = W^+(n,\phi)$ so

$$\lim_{n\to\infty} \frac{W^+(n)}{n} = \alpha^+(\phi)$$

(see Equation (3.12)) and hence

$$\lim_{n\to\infty} \frac{1}{n}\langle W^+(n), n\rangle = \langle \alpha^+(\phi), 1\rangle.$$

It follows that for sufficiently large n, the vector $\langle W^+(n), n\rangle$ is also not parallel to L and points in the direction from the right side of L to the left side. Hence, given any $(u_0, v_0) \in \mathbb{Z}^2$ on the left side of L, there exists $n_0 > 0$ such that if $u_1 = u_0 - W^+(n_0)$ and $v_1 = v_0 - n_0$, then (u_1, v_1) is on the right side of L. The ray $\{(t, v_1) \colon u_1 \le t\}$ in $\mathcal{U}$ lies entirely to the right of L and codes $\{(u_0, v_0)\}$.

It follows that if $x, y \in \mathcal{U}$ agree to the right of L, then they also agree at (u_0, v_0). Since (u_0, v_0) is an arbitrary point to the left of L, it follows that L is left expansive. The proof of (2) is analogous.

To show the final statement, note the the reflection $r \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $r(u, v) = (u, -v)$ has the property that it switches the spacetimes $\mathcal{U}(\phi)$ and $\mathcal{U}(\phi^{-1})$. If L is the line $i = mj$, then $r(L)$ has equation $i = -mj$. We also note that if L is right expansive in $\mathcal{U}(\phi)$, then $r(L)$ is right expansive in $\mathcal{U}(\phi^{-1})$ and vice versa. Similarly $r(L)$ is left expansive in $\mathcal{U}(\phi^{-1})$ if and only if L is left expansive in $\mathcal{U}(\phi)$.

Consider the line L given by $i = mj$ in $\mathcal{U}(\phi)$, and so $r(L)$ is the line $i = -mj$ in $\mathcal{U}(\phi^{-1})$. By part (2), if $-m < \alpha^-(\phi^{-1})$, then the line $r(L)$ is right expansive in $\mathcal{U}(\phi^{-1})$. Equivalently, if $m > -\alpha^-(\phi^{-1})$, then the line $r(L)$ is right expansive in $\mathcal{U}(\phi^{-1})$. Hence $m > -\alpha^-(\phi^{-1})$ implies that L is right expansive in $\mathcal{U}(\phi)$. If we also have $m > \alpha^+(\phi)$ then by part (1), the line L is also left expansive and thus it is, in fact, expansive. The case that $m < \min\{\alpha^-(\phi), -\alpha^+(\phi^{-1})\}$ is handled similarly. □

5. Asymptotic behavior

5.1. **The asymptotic light cone.** The edges of the light cone $\mathcal{C}(\phi)$ are given by the graphs of the functions $i = W^+(j, \phi)$ $i = W^-(j, \phi)$. Since these functions have nice asymptotic properties, so does the cone they determine, which motivates the following definition:

Definition 5.1. The *asymptotic light cone* of ϕ is defined to be

$$\mathcal{A}(\phi) = \{(u, v) \in \mathbb{R}^2 \colon \alpha^-(\phi)v \le u \le \alpha^+(\phi)v\}.$$

This means $\mathcal{A}(\phi)$ is the cone in $\mathbb{R}^2$ which does not contain the i-axis and which is bounded by the lines $u = \alpha^+(\phi)v$ and $u = \alpha^-(\phi)v$. We view $\mathcal{A}(\phi)$ as a subset of $\mathbb{R}^2$ rather than of $\mathbb{Z}^2$, as we want to consider lines with irrational slope that may lie in $\mathcal{A}(\phi)$ but would intersect $\mathcal{C}(\phi)$ only in $\{0\}$.

We begin by investigating the deviation of the function $W^+(n, \phi)$ from the linear function $n\alpha^+(\phi)$. Observe that the asymptotic light cone $\mathcal{A}(\phi)$ is a subset of the light cone $\mathcal{C}(\phi)$, as an immediate corollary of part (3) of Lemma 4.2.

Corollary 5.2. *The set of integer points in the asymptotic light cone $\mathcal{A}(\phi)$ is a subset of the light cone $\mathcal{C}(\phi)$.*

It is natural to consider the relationship between $\mathcal{C}(\phi)$ and $\mathcal{C}(\phi^{-1})$, or between $\mathcal{A}(\phi)$ and $\mathcal{A}(\phi^{-1})$. The spacetime $\mathcal{U}(\phi)$ of ϕ is not the same as the spacetime $\mathcal{U}(\phi^{-1})$ of ϕ^{-1}, but there is a natural identification of $\mathcal{U}(\phi)$ with the reflection of $\mathcal{U}(\phi^{-1})$ about the horizontal axis $j = 0$. In general, it is not true that $\mathcal{A}(\phi^{-1})$ is the reflection of $\mathcal{A}(\phi)$ about the u-axis (Example 2.7 is one where this fails). On the other hand, if (X, σ)

is a subshift of finite type, there is at least one line in the intersection of $\mathcal{A}(\phi^{-1})$ with the reflection of $\mathcal{A}(\phi)$ about the u-axis.

To see this, note that the cone $\mathcal{A}(\phi^{-1})$ has edges which are the lines

$$u = \alpha^+(\phi^{-1})v \text{ and } u = \alpha^-(\phi^{-1})v, \tag{5.1}$$

while the cone obtained by reflecting $\mathcal{A}(\phi)$ about the u-axis has edges given by

(5.2) {eqn: cone edge} $u = -\alpha^-(\phi)v$ and $u = -\alpha^+(\phi)v$.

Hence the line $u = mv$ lies in the intersection $\mathcal{A}(\phi^{-1})$ and the reflection of $\mathcal{A}(\phi)$ in the line u-axis if

$$m \in [\alpha^-(\phi^{-1}), \alpha^+(\phi^{-1})] \cap [-\alpha^+(\phi), -\alpha^-(\phi)].$$

If these two intervals are disjoint, then either

$$\alpha^+(\phi^{-1}) < -\alpha^+(\phi) \text{ or } -\alpha^-(\phi) < \alpha^-(\phi^{-1}).$$

Either of these inequalities contradict part (5) of Proposition 3.13.

In a different vein, the cone $\mathcal{A}(\phi)$ is a conjugacy invariant:

Proposition 5.3. *Suppose (X_i, σ_i) is a shift for $i = 1, 2$ and $\phi_i \in \mathrm{Aut}(X_i)$. Suppose further that $\psi\colon X_1 \to X_2$ is a topological conjugacy from σ_1 to σ_2. If*

$$\phi_2 = \psi \circ \phi_1 \circ \psi^{-1},$$

then $\mathcal{A}(\phi_1) = \mathcal{A}(\phi_2)$.

Proof. Since ψ is a block code, there is a constant $D > 0$, depending only on ψ, such that for any $n \in \mathbb{Z}$ the ray $[n, \infty)$ ψ-codes $[n + D, \infty)$ and the ray $(-\infty, n]$ ψ-codes $(-\infty, n - D]$. It follows that $W^+(m, \phi_1) \le W^+(m, \phi_2) + 2D$. Switching the roles of ϕ_1 and ϕ_2 and considering ψ^{-1}, for which there is $D' > 0$ with properties analogous to those of D, we see that $W^+(m, \phi_2) \le W^+(m, \phi_1) + 2D'$. By the definition of α^+ (see Equation 3.12),

$$\alpha^+(\phi_1) = \lim_{n\to\infty} \frac{W^+(n, \phi_1)}{n} = \lim_{n\to\infty} \frac{W^+(n, \phi_2)}{n} = \alpha^+(\phi_1).$$

The proof that $\alpha^-(\phi_1) = \alpha^-(\phi_2)$ is similar, and thus the asymptotic light cones of ϕ_1 and ϕ_2 are identical. □

5.2. **A partial converse to Theorem 4.4.** In Theorem 4.4 we showed that lines in the spacetime of an automorphism ϕ which form the boundary of its asymptotic light cone $\mathcal{A}(\phi)$ are nonexpansive subspaces. In this section we want to show that in many instances, given an arbitrary $\mathbb{Z}^2$-system Y and a nonexpansive subspace $L \subset \mathbb{R}^2$ for Y, there is a $\mathbb{Z}^2$-system isomorphism Ψ from Y to the spacetime $\mathcal{U}$ of an automorphism $\phi \in \mathrm{Aut}(X, \sigma)$ for some shift (X, σ) such that $\Psi(L)$ is an edge of the asymptotic light cone $\mathcal{A}(\phi)$. In particular this holds if Y has finitely many nonexpansive subspaces. Hence in that case every nonexpansive subspace in Y is (up to isomorphism) an edge of an asymptotic light cone for some automorphism.

To do this it is useful to introduce the notion of *expansive ray* which incorporates both the subspace and the distinction between right and left expansive.

We recall that an ordered basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ is *positively oriented* if the isomorphism of $\mathbb{R}^2$ which maps $\langle 1, 0\rangle$ to e_1 and $\langle 0, 1\rangle$ to e_2 has positive determinant. By a *ray* in $\mathbb{R}^2$ we mean a set $\rho \subset \mathbb{R}^2$ such that there exists $w \neq 0 \in \mathbb{R}^2$ with

$$\rho = \rho(w) = \{tw \colon t \in [0, \infty)\}.$$

The space of all rays in $\mathbb{R}^2$ is naturally homeomorphic to the set of unit vectors in $\mathbb{R}^2$, which is the circle S^1.

Definition 5.4. Let Y be a topological $\mathbb{Z}^2$-system. We say ρ is an *expansive ray* for Y if the line L containing ρ has the property that $H^-_L \cap \mathbb{Z}^2$ codes $H^+_L \cap \mathbb{Z}^2$, where H^+_L is the component of $\mathbb{R}^2 \setminus L$ such that for all $w \in \rho$ and $w' \in H^+_L$ the ordered basis $\{w, w'\}$ is positively oriented and H^-_L is the other component of $\mathbb{R}^2 \setminus L$. If a ray fails to be an expansive ray we call it a *nonexpansive ray.*

The concept of expansive ray is essentially the same as that of *oriented expansive line* introduced in §3.1 of [5]. We emphasize that this concept is defining *one-sided expansiveness* for the line L containing ρ. Which side of L codes the other is determined by the orientation of ρ and the orientation of $\mathbb{Z}^2$.

To relate this to our earlier notions of left and right expansive (Definition 2.5) observe that if L is the span of the vector $w = \langle a, b\rangle \in \mathbb{R}^2$ with $b > 0$, then L is left expansive if and only if the ray $\rho = \{tw \colon t \in [0, \infty)\}$ is an expansive ray and L is right expansive if and only if $-\rho$ is an expansive ray. Hence a line L is expansive if it contains the ray ρ and both ρ and $-\rho$ are expansive rays. In this terminology, Theorem 4.4 says that the rays $\rho^+(\phi) := \{\langle \alpha^+ v, v\rangle \colon v \geq 0\}$ and $\rho^-(\phi) := \{\langle \alpha^- v, v\rangle \colon v \leq 0\}$ are nonexpansive rays. We note that it is not in general the case that $-\rho^+(\phi)$ and $-\rho^-(\phi)$ are nonexpansive rays.

The following result is essentially contained in [3], but differs from results there in that we consider one-sided expansiveness.

Lemma 5.5. *If $\mathcal{E} \subset S^1$ is the set of expansive rays for a topological $\mathbb{Z}^2$-system Y, then $\mathcal{E}$ is open.*

Proof. We show that the set $\mathcal{N}$ of nonexpansive rays is closed. Suppose that $\rho_n = \{tw_n \colon t \geq 0\}_{n=1}^{\infty}$ is a sequence of rays in $\mathbb{R}^2$ with $\lim_{n\to\infty} w_n = w_0 \neq 0$ so that ρ_0 is the limit of the rays $\rho_n, n \geq 1$. If the rays ρ_n are nonexpansive we must show that ρ_0 is nonexpansive.

Let L_n be the line containing w_n and let $H^+_{L_n}(w_n)$ be the component of $\mathbb{R}^2 \setminus L_n$ such that for all $w' \in H^+_{L_n}$ the ordered basis $\{w_n, w'\}$ is positively oriented and let $H^-_{L_n}(w_n)$ be the other component of $\mathbb{R}^2 \setminus L_n$. Define the linear function $f_n \colon \mathbb{R}^2 \to \mathbb{R}$ by $f_n(u) = u \cdot v_n$ where v_n is a unit vector in $H^+_{L_n}$ which is orthogonal to w_n. Then we have the following:

- $L_n = \ker(f_n)$
- A vector u is in $H^+_{L_n}$ if and only if $f_n(u) > 0$ and in $H^-_{L_n}$ if and only if $f_n(u) < 0$.
- $\lim_{n\to\infty} f_n(v_0) = f_0(v_0) = 1.$

By the definition of expansive (2.5) we know there exist $\eta_n, \eta'_n \in Y$ and $z_n \in \mathbb{Z}^2$ such that $\eta_n(v) = \eta'_n(v)$ for all $v \in H^-_{L_n}$ but $\eta_n(z_n) \neq \eta'_n(z_n)$. By shifting η_n and η'_n we may assume lengths $|z_n|$ are bounded. Choosing a subsequence we may assume $\{z_n\}$ is constant, say, $z_n = z_0 \in Z^2$. Since Y is compact we may further choose subsequences $\{\eta_n\}_{n=1}^\infty$ and $\{\eta'_n\}_{n=1}^\infty$ which converge, say to η_0 and η'_0 respectively. Clearly $\eta_0(z_0) \neq \eta'_0(z_0)$. Now if $y \in H^-_{L_0} \cap \mathbb{Z}^2$ then $f_0(y) < 0$ so $f_n(y) < 0$ for sufficiently large n and hence $y \in H^-_{L_n} \cap \mathbb{Z}^2$. It follows that $\eta_0(y) = \eta'_0(y)$.

Since η_0 and η'_0 agree on $H^-_{L_n} \cap \mathbb{Z}^2$ but disagree at z_0 we conclude that $H^-_{L_n} \cap \mathbb{Z}^2$ does not code $H^+_{L_n} \cap \mathbb{Z}^2$ so ρ_0 is a nonexpansive ray. □

Proposition 5.6. *Suppose Y is a topological $\mathbb{Z}^2$-system and $\mathcal{E}$ is the set of expansive rays for Y (thought of as a subset of S^1). Suppose C is a component of $\mathcal{E}$ and ρ_1, ρ_2 are the endpoints of the open interval C. Then there exists a shift (X, σ) with automorphism $\phi \in \mathrm{Aut}(X)$ and an isomorphism $\Psi\colon Y \to \mathcal{U}(\phi)$ from Y to the spacetime of ϕ such that the lines $L_1 := \mathrm{span}(\Psi(\rho_1))$ and $L_2 := \mathrm{span}(\Psi(\rho_2))$ are the two edges of the asymptotic light cone $\mathcal{A}(\phi)$ of ϕ.*

Proof. We consider C as an open interval (ρ_1, ρ_2) in the circle S^1 of rays in $\mathbb{R}^2$. There is a $\mathbb{Z}^2$-system isomorphism $\Psi_0\colon Y \to Y_0$, where Y_0 is $\mathbb{Z}^2$-system with $\langle 1, 0\rangle \in \Psi_0(C)$. Thus the horizontal axis is an expansive subspace for the $\mathbb{Z}^2$-system Y_0. We may recode Y_0 to Y_1 by an isomorphism $\Psi_1\colon Y_0 \to Y_1$ such that the horizontal axis H_0 in $\mathbb{Z}^2$ codes the positive half space $\{\langle i, j\rangle \in \mathbb{Z}^2 \colon j > 0\}$ (this follows from Lemma 3.2 in [3] where we recode Y_0 such that "symbols" in Y_1 are vertically stacked arrays of symbols from Y_0 of an appropriate height). We let $\Psi\colon Y \to Y_1$ be the composition $\Psi_1 \circ \Psi_0$.

Let X denote the set of colorings of $\mathbb{Z}$ obtained by restricting elements $\eta \in Y_1$ to H_0. We could equally well describe X as the colorings of $\mathbb{Z}$ obtained by restricting elements of Y to the horizontal row $H_{-1} := \{\langle i, j\rangle \in \mathbb{Z}^2 \colon j = -1\}$ and define $\phi\colon X \to X$ by $\phi(x) = x'$ if there is $\eta \in Y_1$ such that $x = \eta|_{H_0}$ and $x' = \eta|_{H_{-1}}$. Then clearly $\phi \in \mathrm{Aut}(X)$ and Y_1 is $\mathcal{U}(\phi)$, the spacetime of ϕ.

Note that the ray $\rho^+(\phi) := \{\langle \alpha^+ v, v\rangle \colon v \geq 0\}$ lies in the light cone $\mathcal{A}(\phi)$ of ϕ (and in the upper half space of $\mathbb{R}^2$). If $m > \alpha^+(\phi)$ and ρ_m is the ray $\rho_m := \{\langle mv, v\rangle \colon v \geq 0\}$, then by Proposition 4.5 ρ_m is an expansive ray. Since by Theorem 4.4 $\rho^+(\phi)$ is not an expansive ray, it follows that $\Psi(\rho_2) = \rho^+(\phi)$.

Letting $\rho^-(\phi) := \{\langle \alpha^- v, v\rangle \colon v \leq 0\}$, a similar proof shows that $\Psi(\rho_1) = \rho^-(\phi)$. Hence the lines L_1 and L_2 form the edges of the asymptotic light cone $\mathcal{A}(\phi)$. □

We are not able to show which lines can arise as the edges of the asymptotic light cone:

Question 5.7. *Does there exist a subshift of finite type X and an automorphism $\phi \in \mathrm{Aut}(X)$ such that some edge of the asymptotic light cone of ϕ has irrational slope? If so, what set of angles is achievable?*

Hochman [12] points out that, as there are only countably many shifts of finite type, this set must be countable (and, in particular, cannot all have irrational slopes).

5.3. **Asymptotic spread.** Let $r(n, \phi)$ be the minimal length of an interval $J \subset \mathbb{Z}$ which contains 0 and ϕ^n-codes $\{0\}$ and let $\operatorname{range}(\phi^n)$ be the minimal length of an interval $J_0 \subset \mathbb{Z}$ which is symmetric about 0 and ϕ^n-codes $\{0\}$. It is straightforward to see that both $r(n, \phi)$ and $\operatorname{range}(\phi^n)$ are subadditive sequences.

Definition 5.8. Define the *asymptotic spread* $A(\phi)$ of $\phi \in Aut(X)$ to be

$$A(\phi) = \lim_{n\to\infty} \frac{r(n, \phi)}{n}. \tag{5.3}$$

We say $\phi \in \operatorname{Aut}(X)$ is *range distorted* if $A(\phi) = 0$.

Note that since the sequence $r(n, \phi)$ is subadditive, Fekete's Lemma implies that the limit in (5.3) exists.

The asymptotic spread is a measure of both the width of the asymptotic light cone, as well as how that cone deviates from the vertical.

Remark 5.9. By Fekete's Lemma, the limit

$$\rho(\phi) = \lim_{n\to\infty} \frac{\operatorname{range}(\phi^n)}{n}$$

exists. Clearly $\operatorname{range}(\phi^n) \le r(n, \phi) \le 2\operatorname{range}(\phi^n) + 1$ and so

$$\rho(\phi) \le A(\phi) \le 2\rho(\phi).$$

In particular, ϕ is range distorted if and only if

$$\lim_{n\to\infty} \frac{\operatorname{range}(\phi^n)}{n} = 0$$

Proposition 5.10. *If $\phi \in \operatorname{Aut}(X)$ and $\alpha^+(\phi) = \alpha^-(\phi) = \alpha^+(\phi^{-1}) = \alpha^-(\phi^{-1})$, then the line $u = \alpha^+(\phi)v$ is the unique nonexpansive one-dimensional subspace. In particular, if $\phi, \phi^{-1} \in \operatorname{Aut}(X)$ are both range distorted, then the vertical axis ($u = 0$) is the unique nonexpansive subspace*

Proof. The first statement follows immediately from Theorem 4.4 and Proposition 4.5. The second statement follows from the first, since ϕ and ϕ^{-1} are both range distorted if and only if $\alpha^+(\phi) = \alpha^-(\phi) = \alpha^+(\phi^{-1}) = \alpha^-(\phi^{-1}) = 0$. □

It was shown by M. Hochman [12] that if L is any 1-dimensional subspace of $\mathbb{R}^2$, then there exists a subshift X_L and an automorphism $\phi_L \in \operatorname{Aut}(X_L)$ such that L is the unique nonexpansive subspace for the spacetime of ϕ_L. Moreover, the automorphisms ϕ_L in his examples always have infinite order (in particular, when L is vertical, ϕ_L is range distorted and has infinite order). However, the space X_L he constructs lacks many natural properties one might assume about a subshift; for example, it is not a subshift of finite type and it is not transitive. He asks the following natural question:

Question 5.11 (Hochman [12, Problem 1.2])**.** *Does every nonempty closed set of one-dimensional subspaces of $\mathbb{R}^2$ arise as the nonexpansive subspaces of a $\mathbb{Z}^2$-action that is transitive (or even minimal) and supports a global ergodic measure?*

We do not answer this question, but recall it here as, in particular, we do not know whether a transitive subshift can have a range distorted automorphism of infinite order. We mention further that, in the special case that L is vertical, Hochman shows that his example (X_L, ϕ_L) is logarithmically distorted.

Proposition 5.12. *If ϕ is an automorphism of a subshift of finite type (X, σ), then $A(\phi)$ is determined by the light cone of ϕ and is, in fact, the length of the smallest interval containing $0, \alpha^-(\phi)$ and $\alpha^+(\phi)$.*

Proof. It follows from Proposition 3.19 that if σ is a subshift of finite type, then for all $x \in X$ and all sufficiently large $n > 0$, the interval $[W^-(n), W^+(n)]$ is an interval which codes $\phi^n(x)[0]$ and that no smaller interval suffices. It follows that if J_n is the smallest interval containing $0, W^-(n)$ and $W^+(n)$, then

$$A(\phi) = \lim_{n\to\infty} \frac{|J_n|}{n}.$$

Hence $A(\phi)$ is the length of the smallest interval containing $0, \alpha^-(\phi)$ and $\alpha^+(\phi)$. □

The following result is essentially the same as Proposition 5.3 of Tisseur's paper [20], except that we consider an arbitrary $\phi \in \mathrm{Aut}(X, \sigma)$ with σ an arbitrary shift while he considers a cellular automaton defined on the full shift and preserving the uniform measure on that shift. Our proof is quite short and makes no use of measure. It makes explicit the connection between the topological entropy of a shift and the topological entropy of an automorphism of that shift.

Theorem 5.13. *If $\phi \in \mathrm{Aut}(X)$, then*

$$h_{\mathrm{top}}(\phi) \le A(\phi) h_{\mathrm{top}}(\sigma),$$

where $A(\phi)$ is the asymptotic spread of ϕ. In particular, if ϕ is range distorted then $h_{\mathrm{top}}(\phi) = 0$.

Proof. Let $\mathcal{U}$ be the spactime of ϕ. For $z \in \mathcal{U}$, let $R_{m,n} = \{(i,j) \in \mathbb{Z}^2 \colon 0 \le i < m, 0 \le j < n\}$ and let $z|_{R_{m,n}}$ denote the restriction of z to $R_{m,n}$. Recall that $P_{\mathcal{U}}$ denotes the two dimensional complexity function $P_{\mathcal{U}}$ (see Definition 2.4). Then

$$h_{\mathrm{top}}(\phi) = \lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{n} \log(P_{\mathcal{U}}(R_{m,n})).$$

Since $A(\phi)$ is the length of the smallest interval containing $0, \alpha^-(\phi)$ and $\alpha^+(\phi)$, for a fixed m there is an interval J in $\mathbb{Z}$ with length $A(\phi)n + \mathrm{o}(n) + m$ that ϕ^j-codes the block $[0, m]$ for all $0 \le j \le n$. In other words, the interval $J \times \{0\} \subset \mathcal{U}$ codes $R_{m,n}$. Therefore, for any $\varepsilon > 0$, and m and n sufficiently large,

$$P_{\mathcal{U}}(R_{m,n}) \le P_X(A(\phi)n + \mathrm{o}(n) + m) \le (\exp(h_\sigma + \varepsilon))^{A(\phi)n+m}.$$

Hence $\log(P_{\mathcal{U}}(R_{m,n})) \le (A(\phi)n + m)(h_\sigma + \varepsilon)$ and

$$h_{\mathrm{top}}(\phi) = \lim_{m\to\infty} \lim_{n\to\infty} \frac{\log(P_{\mathcal{U}}(R_{m,n}))}{n} \le \lim_{m\to\infty} \lim_{n\to\infty} \frac{(A(\phi)n + m)(h_\sigma + \varepsilon)}{n} = A(\phi)(h_\sigma + \varepsilon).$$

Since this holds for all $\varepsilon > 0$, the desired inequality follows.

By definition ϕ is range distorted if and only if $A(\phi) = 0$, and so the last two assertions of the proposition are immediate. □

5.4. **Distortion and inert automorphisms.** Recall that if (Σ_A, σ) is a subshift of finite type, there is a dimension group representation $\Psi\colon \mathrm{Aut}(\Sigma_A) \to \mathrm{Aut}(D_A)$ mapping automorphisms of the shift to automorphisms of its dimension group D_A (see [16], [21], and [1] for definitions). A particularly important subgroup of $\mathrm{Aut}(\Sigma_A)$ is $\mathrm{Inert}(\Sigma_A)$, defined to be the kernel of Ψ. An automorphism $\phi \in \mathrm{Aut}(\Sigma_A)$ is called *inert* if $\Psi(\phi) = \mathrm{Id}$.

There is one special case when Ψ can be thought of as a homomorphism from $\mathrm{Aut}(\Sigma_A)$ to the group of positive reals under multiplication. This occurs when Σ_A is an irreducible subshift of finite type and $\det(I - At)$ is an irreducible polynomial. In this setting, one can associate to each $\phi \in \mathrm{Aut}(\Sigma_A)$ an element $\lambda_\phi = \Psi_0(\phi)$ in $(0, \infty)$ such that Ψ_0 is a homomorphism and $\lambda_\phi = 1$ if and only if ϕ is inert.

To investigate the relationship between being inert and being distorted, we quote the following important result of Boyle and Krieger:

Theorem 5.14 (Boyle and Krieger [1, Theorem 2.17])**.** *Suppose (Σ_A, σ) is an irreducible subshift of finite type and $\det(I - At)$ is an irreducible polynomial. Then if $\phi \in \mathrm{Aut}(\Sigma_A)$ and m is sufficiently large, $\sigma^m\phi$ is conjugate to a subshift of finite type and*

$$h_{\mathrm{top}}(\sigma^m\phi) = \log(\lambda_\phi) + m h_{\mathrm{top}}(\sigma).$$

Theorem 5.15. *Suppose (Σ_A, σ) is an irreducible subshift of finite type such that $\det(I - At)$ is an irreducible polynomial, and let $\phi \in \mathrm{Aut}(\Sigma_A)$. If ϕ and ϕ^{-1} are range distorted, then ϕ is inert.*

Proof. Let $\lambda_\phi = \Psi(\phi)$ and note that by replacing ϕ with ϕ^{-1} if necessary, we can assume that $\lambda_\phi \geq 1$. Suppose ϕ is range distorted and so $\alpha^+(\phi) = \alpha^-(\phi) = 0$; we show that ϕ is inert. From parts (1) and (2) of Proposition 3.13, we conclude that $\alpha^+(\sigma^k\phi) = \alpha^-(\sigma^k\phi) = k$. By Proposition 5.12, it follows that $A_{\sigma^k\phi} = |k|$. Hence by Theorem 5.13, we have $h_{\mathrm{top}}(\sigma^k\phi) \leq |k| h_{\mathrm{top}}(\sigma)$. Combining this with the fact from Theorem 5.14 which says for large k we have $h_{\mathrm{top}}(\sigma^k\phi) = \log(\lambda_\phi) + k h_{\mathrm{top}}(\sigma)$, we conclude that $\log(\lambda_\phi) \leq 0$ or $\lambda_\phi \leq 1$. Since we also have $\lambda_\phi \geq 1$, we conclude that $\lambda_\phi = 1$ and ϕ is inert. □

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